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ABSTRACTS

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Plenary lectures

Moonstruck – The Interplay of Celestial Bodies in Pictures

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The moon has always held a great fascination for us humans. This is probably because its complex orbit brings it closer to the Earth than any other celestial body. The similarities and differences to other planets and their moons, its interactions with the Sun and the Earth, interesting historical anecdotes - all this is presented in this talk to geometrically experienced and astronomically interested participants with many photographs and computer simulations in concise texts.

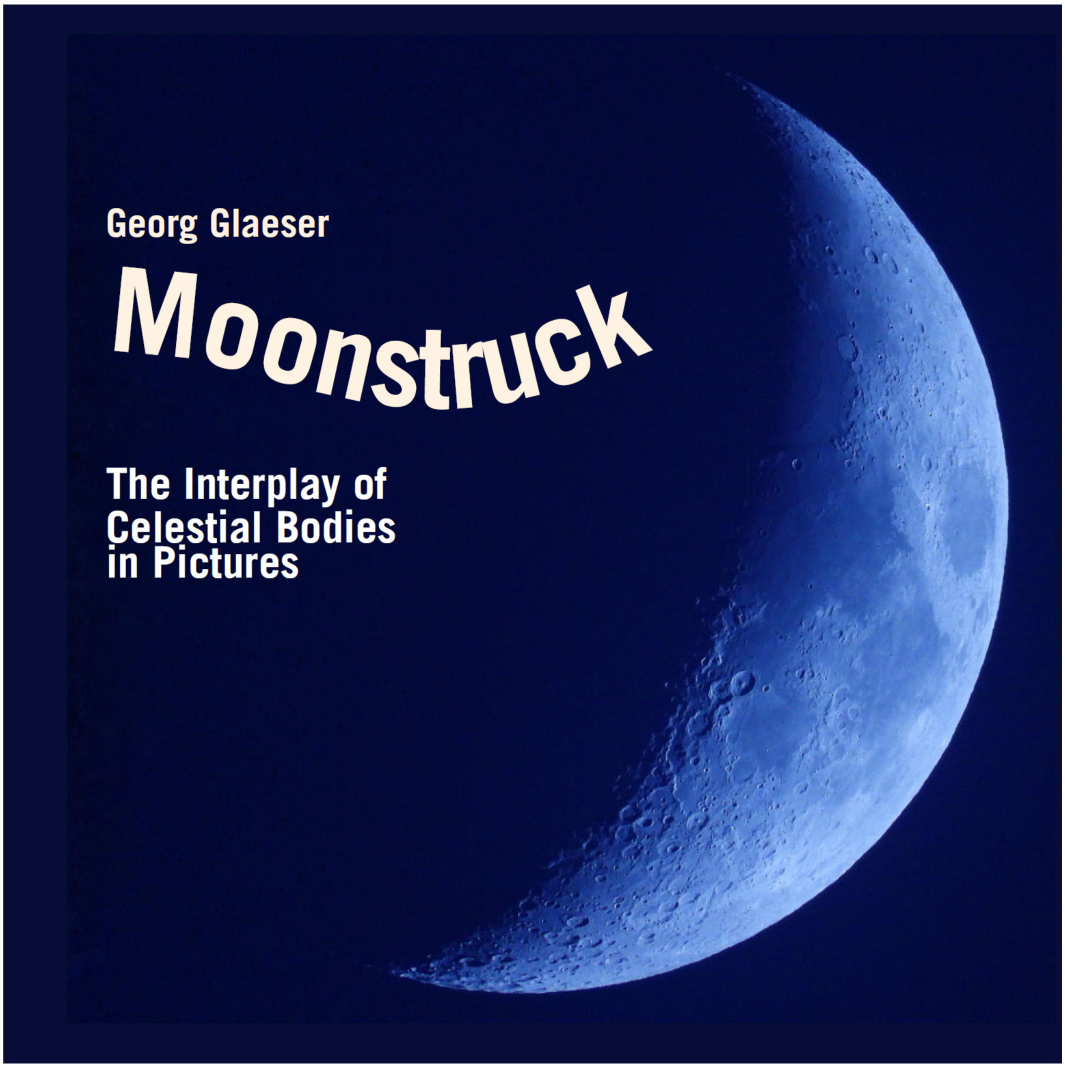

Figure 1: Title of the lecture and of the author's book of the same name

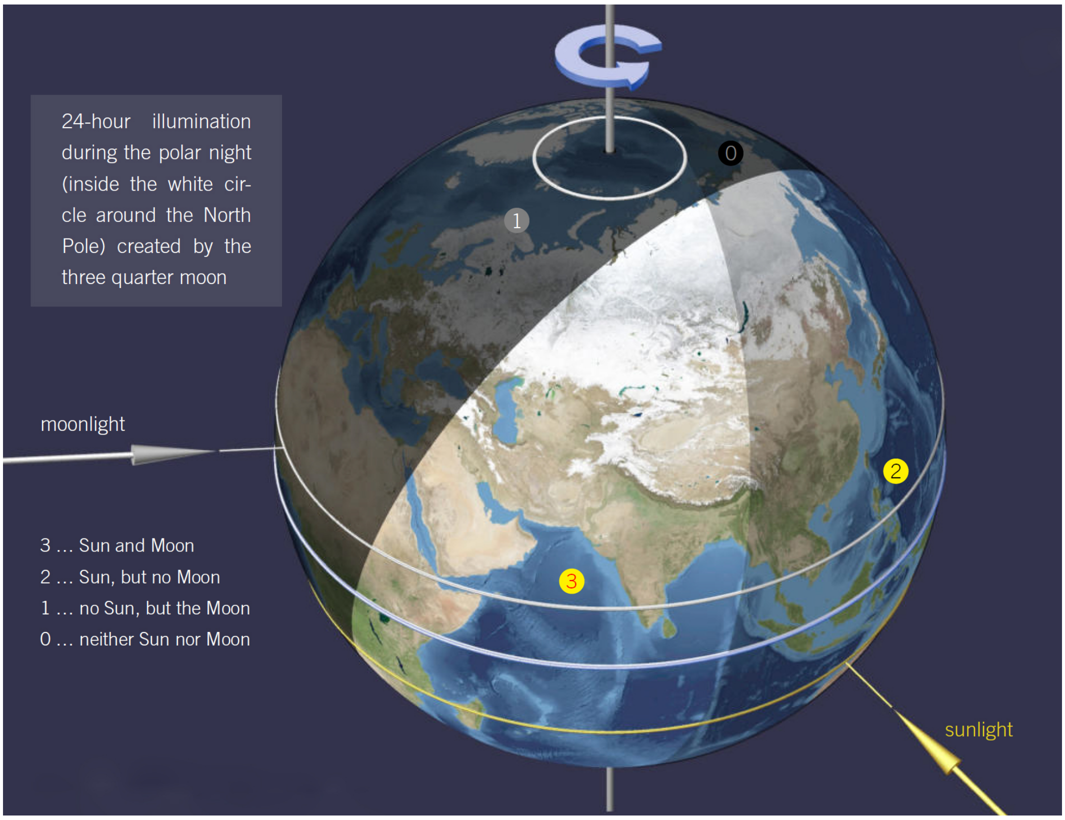

Figure 2: Both the Sun and the Moon illuminate the Earth

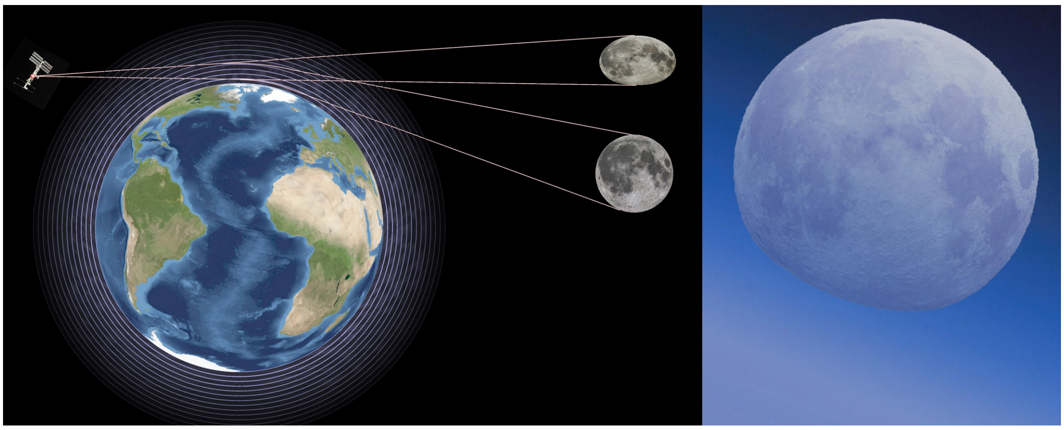

Figure 3: Refraction of moonlight at the Earth's atmosphere

Special Entirely Spherical Surfaces in Euclidean Space

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This is the joint work with Ema Jurkin.

Entirely spherical surfaces in Euclidean space are the surfaces that contain the absolute conic of the space as the curve with the highest multiplicity. In homogeneous coordinates (x, y, z, w), where $w = 0, 1$, and the points at infinity are determined by $w = 0$, the absolute conic is given by the following equations

$$A_2 = x^2 + y^2 + z^2 = 0, \qquad w = 0.$$

In paper [1] we show that entirely spherical surfaces, the surfaces of order $2n$ which contain the absolute conic as an $n-$fold curve, are given by the following affine equation

$$A_2^n + \sum_{j=1}^{n-1} A_2^{n-j} g_j(x, y, z) + \sum_{i=0}^{n} f_i(x, y, z) = 0, \tag{1}$$

where $f_n \neq 0$ and $A_2 \nmid f_n$.

In the mentioned paper we give examples of entirely spherical surfaces with one and two real $n-$fold points. For the construction of these surfaces we introduce a class of entirely circular curves k^{2n}, of order $2n$ and with $n-$fold point at the origin, given by equations

$$\left(x^2 + y^2\right)^n + f_n(x, y) = 0, \tag{2}$$

$$f_n = \begin{cases} \prod_{i=0}^{n-1} \left(\cos i\frac{2\pi}{n} \cdot y - \sin i\frac{2\pi}{n} \cdot x\right), & n \quad \text{odd}, \\ \prod_{i=0}^{n-1} \left(\cos i\frac{\pi}{n} \cdot y - \sin i\frac{\pi}{n} \cdot x\right), & n \quad \text{even}. \end{cases}$$

According to [2], the circular surface $\mathcal{CS}(\alpha, p)$, where $\alpha = k^{2n}$, $p \in \mathrm{R}$ and $p \neq 0$, is an entirely spherical surface of order $2n$ with two real $n-$fold points. Based on [2] we are able to obtain the parametric equations of these surfaces and visualize their shapes. Here, starting with the polar equations of curves k^{2n} and a simpler construction, we derive the following implicit equations of such surfaces

$$A_2^n + \sum_{j=1}^{n-1} \binom{n}{j} (-pz)^j A_2^{n-j} - \mathcal{T}^n(x, y, z) = 0, \tag{3}$$

$$\mathcal{T}^n(x, y, z) = \mathcal{G}^n(x, y) - (-pz)^n,$$

$$\mathcal{G}^n(x,y) = \begin{cases} (-1)^{\frac{n-1}{2}} \left(\sqrt{x^2+y^2}\right)^n T_n\left(\frac{y}{\sqrt{x^2+y^2}}\right), & n \quad \text{odd}, \\ (-1)^{\frac{n}{2}-1} x \left(\sqrt{x^2+y^2}\right)^{n-1} U_{n-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right), & n \quad \text{even}, \end{cases}$$

where T_n and U_n are Chebyshev polynomials of the first and second kind, respectively. According to [3], $\mathcal{T}^n(x,y,z)=0$ is the equation of the tangent cone at the origin which is the $n-$fold point of the surface.

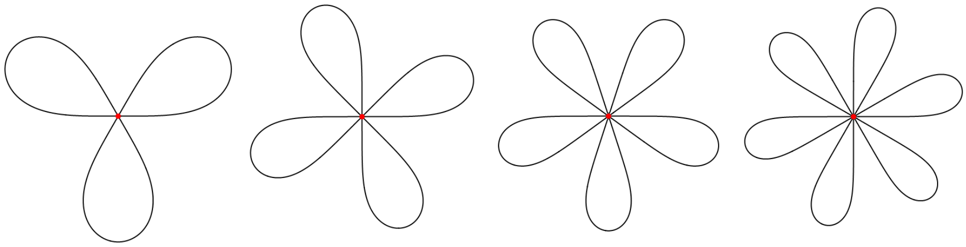

Figure 1: Entirely circular curves k^{2n} given by equation (2) for $n = 3, 4, 5, 6$.

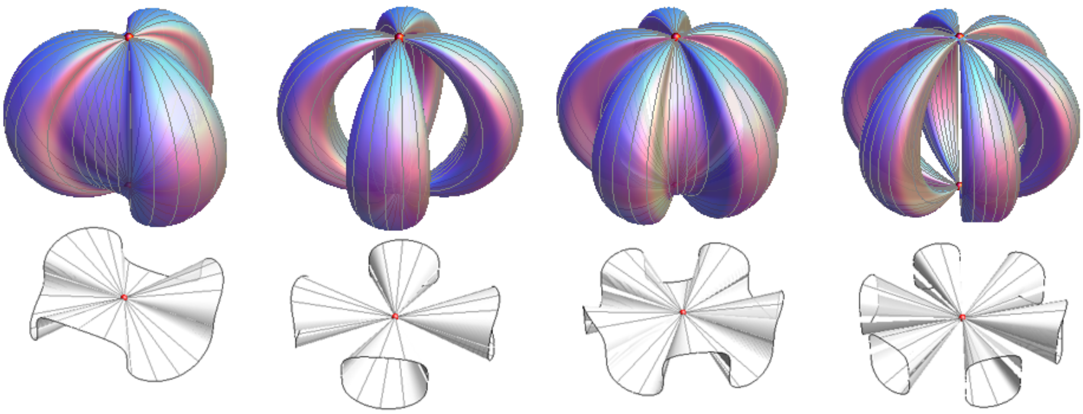

Figure 2: Entirely spherical surfaces given by equation (3) for $n = 3, 4, 5, 6$, and their tangent cones at the origin.

Key words: entirely circular curve, entirely spherical surface

MSC 2010: 51N20, 51M15

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Exploration of Measure

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"Es ist die Freude an der Gestalt in einem höheren Sinne, die den Geometer ausmacht." (Clebsch, in memory of Julius Plücker, Göttinger Abh. Bd. 15), [4].

All notions have a name. The name itself, if appropriate, carries the metaphysical meaning of the notion. Philosophy starts and ends with Φ, the most popular number, honored by a name referring to the highest value - the *golden* number. We can ask, what is this value that was referred to when the name had been given? What are the standards that this number so richly satisfies? By studying interpretations, constructions and structures related to this number, either numerical or geometrical, we find regularities of a fractal nature that carry mathematical depth, inner richness and account for its great value. We can then understand why it is called the golden section, proportion, ratio, mean. Here in this simple observation of a single number, we see the richness of the notion of measure. In the most abstract sense, notion of measure best abridges to *quantity of quality*, where both terms are adaptable to any given situation. Setting standards for both quality and quantity is the determination of a value structure of a given observed system.

The intention of this talk is to discuss historical dealings with the notion of measure in geometry and its implications and relations to the contemporary zeitgeist.

The term geometry is a word coined to represent the universal concepts of measurement and regularity by the Greeks living in ancient Egypt. Each year, the river Nile flooded the fields of the Egyptian farmers and each year after the flood they had to determine the borders of their lands anew. They had some fractal information before the act of measurement and the way they managed to create new divisions in a manner that satisfied the majority of farmers and kept the peace is what thrilled and attracted the attention of Greek mathematicians who ultimately saw a universal regularity in the entirety of this process of measurement. This was the influence behind the Elements, the first known deductive exposition of geometry, written by Euclid, who was active in Alexandria around 3^{rd} century BC.

The deductive system and Euclid's observations of the universal regularity of forms and structures play an indispensable role in the technological progress of humankind. The historical developments on the conception of the nature of space and well as the meaning and application of measure done at the University of Göttingen in 19^{th} and early 20^{th} century, starting with Gauss and Riemann, then Hilbert, Klein, Minkowski and Einstein led to the creation of binary machines that comprise the basis of modern technology of our current reality, where prominent research in geometry is mostly intended for and applied in computer science, "*digitalized geometry*", as is happening to many other disciplines.

But given the purpose, if geometry is the study of spatial relations, what is this digital space and where can it lead us scientifically? The discreteness of the nature of binary machines we use cannot store, comprehend or transfer in any way the transcendental nature of geometry that surrounds us. There is literally no room for Φ there, in its fullness. If this is the basis of our observations, are the higher dimensions, *den höheren Sinne, die der Geometrie ausmacht* lost? Where can the intention and purpose to study geometry be found?

Figure 1: *Go figure!*, my recent acrylic painting representing the number $\Phi = 2\cos 36°$

Key words: measure, philosophy, geometry

MSC 2010: 00A30, 51P99, 86A30

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Spatial Thinking

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At the begin of the 20th century, the ability to think spatially was recognized as one of the crucial facets of human intelligence and intensive research has been carried out in this area by researchers from different branches of science since then. In this talk the four relevant areas of science that deal with spatial thinking in depth will be presented [1-5]. The different approaches and models are discussed, which opens the possibility of being able to see the topic of spatial thinking in a more comprehensive way. Fully understanding the historical and scientific context of spatial thinking research will provide a solid basis for further developments in this area. Also, references are made to the spatial thinking training platform RIF2.0 and to the model of "the basic practices of spatial thinking", which was developed with the intention of extracting the essence from the findings of the four scientific areas and to bring them together in one model.

Key words: spatial thinking, spatial ability, spatial perception, geometry education

MSC 2010: 97G99, 91E30

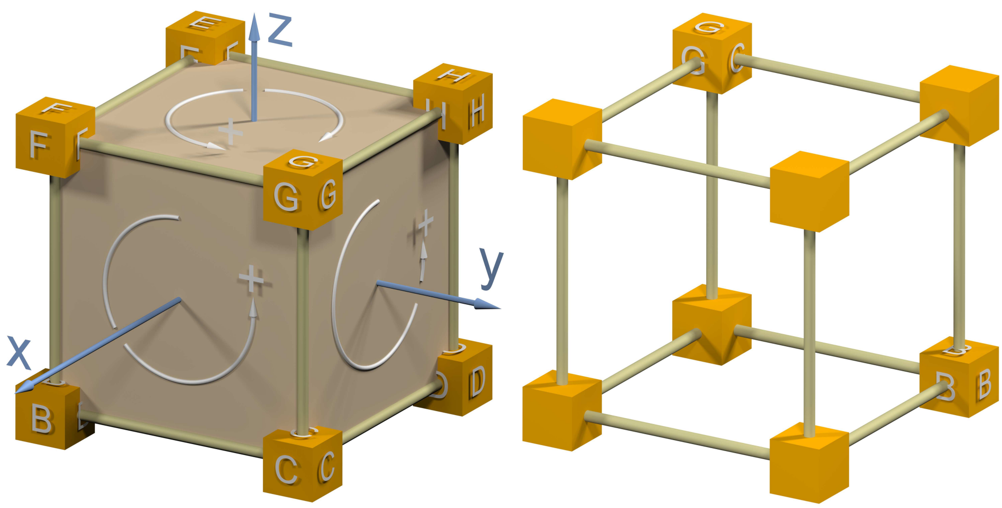

Figure 1: A Spatial Thinking Task from RIF 2.0 (`https://adi3d.at/rif20/en/`)

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3D Generalization of Simson–Wallace Theorem on Four Lines

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Motivation to this problem arose from the Wallace-Simson theorem which states, that feet of perpendiculars from a point P to three lines are collinear if and only if the point P belongs to the circumcircle of the triangle given by these three lines. 3D generalization of the Wallace-Simson theorem might be as follows: Determine the locus of the point P such that feet of normals from P to four arbitrary straight lines in three-dimensional Euclidean space are coplanar.
Generally, the locus is a cubic surface, see [1, 2, 3, 4]. We investigate a special case of straight lines being parallel to a fixed plane. This gives an interesting result — a cylinder of revolution. Furthermore we explore another positions of the given straight lines, for instance the case when the lines form a skew quadrilateral. We also study the positions of the given lines when the locus is a quadric.

Key words: Wallace-Simson theorem, cubics, quadrics, elimination

MSC 2010: 51Mxx, 51N20, 51N35

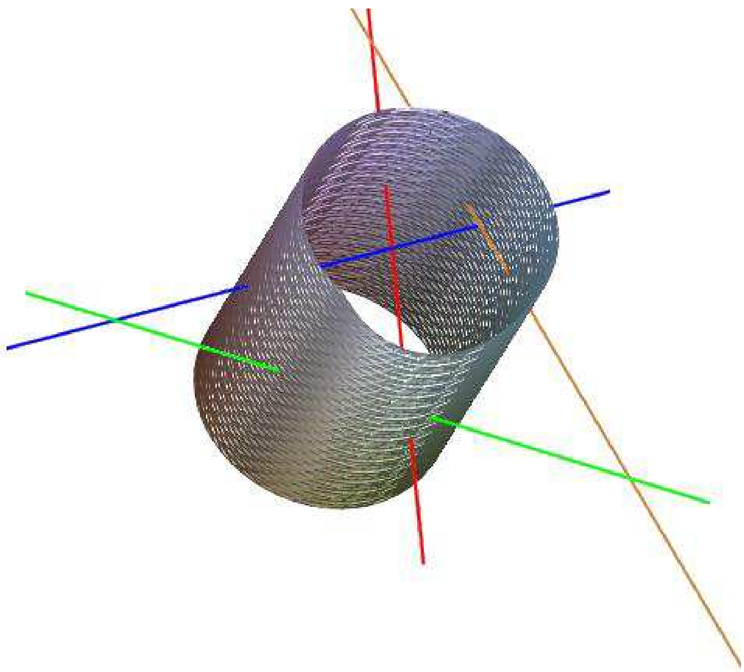

Figure 1: If the lines are parallel to a fixed plane the locus is a cylinder of revolution

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Contributed talks

Reflection Techniques in Real-Time Computer Graphics

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Reflections have a long history in computer graphics, as they are very important for conveying a sense of realism as well as depth and proportion. Their implementations come with a multitude of difficulties, and each solution typically has various trade-offs. Approaches highly depend on the geometry of the reflective surface since curved reflectors are usually more difficult to portray accurately. Techniques can typically be categorized by whether they work with the actual geometry of the reflected objects or with an image of them. For curved surfaces image-based techniques are usually preferred, whereas for flat surfaces the reflected geometry can be used more easily because of the lack of distortion. With current advances in graphics hardware technology, ray tracing is also becoming more viable for real-time applications. Many modern solutions often combine multiple approaches to form a hybrid technique.

In this paper we give an overview of the techniques used in computer graphics applications to achieve real-time reflections. We highlight the trade-offs that have to be dealt with when choosing a particular technique as well as their ability to produce interreflections. Further, we cover how contemporary state-of-the-art rendering engines deal with reflections. Finally, we describe an exemplary implementation of a geometry-based reflection technique which we used to visualize the geometric structure of a fullerene.

Keywords: reflections, interreflections, real-time rendering

MSC 2010: 51-04, 51P05, 78A05

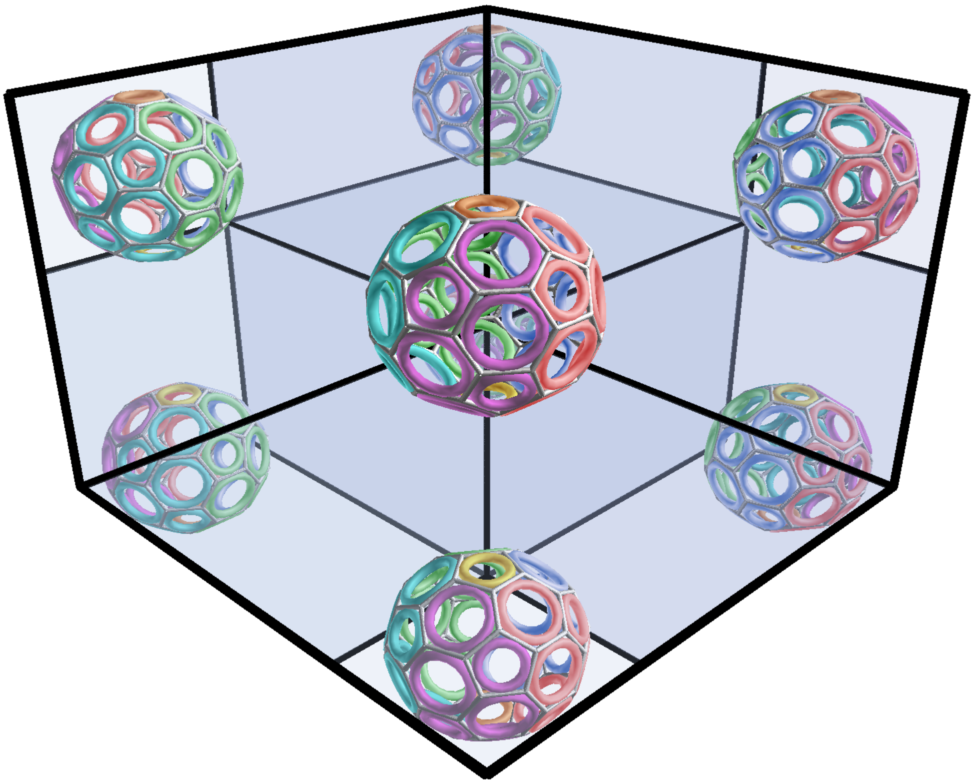

Figure 1: Multiple reflections of a C60 fullerene using a geometric reflection technique

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Reliability Measure of Online Assessment within Mathematics

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Due to the COVID–19 pandemic the educational institutions around the world have been temporarily closed by their governments which resulted in rapid transition to remote learning and online assessment. We critically examine its role in acquisition of knowledge, skills and attitudes. We advocate construction of the complete assessment programme with carefully prepared assessment set, but also its evaluation according to an overarching framework. For that purpose we have chosen the so-called utility framework introduced by Van der Vleuten & Schurwith, 2005.

We will present the utility framework which consists of five elements (validity, reliability, educational impact, acceptability and costs). The focus will be on the reliability which we broadly understand as the accuracy of pass and fail decisions. More precisely, we will introduce and analyse several approaches for construction of the composite index for the assessment programme on the course level as a possible measure of the reliability. To do that, we use data available in the learning management system for past two academic years.

Key words: e-assessment, online test, utility, reliability, measure

MSC 2010: 97D60, 97B40, 97C70, 97–06

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Counting Perfect Stars

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We discuss what makes an n-pointed star perfect, for which number of points such stars exist, and how many of them there are.

Generalizations of Killing Vector Fields in Sol Space

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Killing vector field on Riemannian manifold (M, g) is a vector field X which satisfies the Killing equation $\mathfrak{L}_X g = 0$, where $\mathfrak{L}$ denotes a Lie derivative. The Killing equation expresses that a metric of Riemannian manifold is invariant under the vector field X. Killing vector field flows preserve shapes and sizes and they are manifestations of symmetries in the context of general relativity.

We consider the generalizations of the Killing vector fields in the 3D Sol space. Conformal Killing vector fields are the first generalization. They are defined by the conformal Killing equation $\mathfrak{L}_X g = \lambda g$, where λ is a smooth function on M. 2-Killing vector fields defined by the 2-Killing equation $\mathfrak{L}_X(\mathfrak{L}_X g) = 0$ are the second generalization.

We characterize proper conformal Killing vector fields and determine some proper 2-Killing vector fields in Sol space. It seems that there are no proper conformal Killing vector field in Sol space. On the other hand, we explore the proper 2-Killing vector fields in Sol space and it seems that the approach used here can be starting point for classification of 2-Killing vector fields in other 3D homogeneous Riemannian geometries.

Key words: Killing vector field, Sol space, conformal vector field, Lie derivative

MSC 2010: 53C30, 53C80

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Involute of Pseudo-Null Curve in Minkowski Space

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An involute of a curve in space is a curve to which all tangent lines of the given curve are normal. It is also known for the property that it can be realized as the locus of the free end of a taut string that is unwound from the initial curve. A curve possesses a one-parameter family of involutes and they are all parallel. Involutes of a curve c parametrized by arc-length are given by

$$i(s) = c(s) + (-s + a)t(s),$$

where a is a constant, and $t = c'$.

In the Lorentzian setting, the involute and evolute curves of the spacelike curve with non-null normals have been investigated in [1, 2], of the timelike curve in [3] and of null curve in [4]. The involute and evolute curves of the pseudo-null curves, that is, spacelike curves with null principal normal, have been investigated in [5], where it is stated that involutes of pseudo-null curves do not exist. In this presentation we correct this result and we investigate properties of involute of pseudo-null curve in 3-dimensional Minkowski space.

Key words: Involute, Minkowski space, pseudo-null curve

MSC 2010: 53A10, 53B30

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On Normals of Ellipses and Ellipsoids

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Distance computation is an inportant issue with applications to path planning, obstacle recognition and collision prevention. To compute distances between complex objects these objects are often decomposed into elementary and preferably convex components such as spheres and ellipsoids. Locally extremal distances between such objects occur on their common normals. In this presentation we discuss the task of finding all common normals between basic object pairs like point - ellipsoid, straight line - ellipsoid and ellipsoid - ellipsoid. For each of these pairs we present geometric proofs for the maximal number of common normals in case of generic relative position of the two objects. To that end we use tools from descriptive geometry, line geometry and algebraic geometry.

Key words: ellipse, ellipsoid, evolute, normal congruence, completely circular algebraic curve, Bézout's theorem, Halphen's theorem

MSC 2010: 51-08

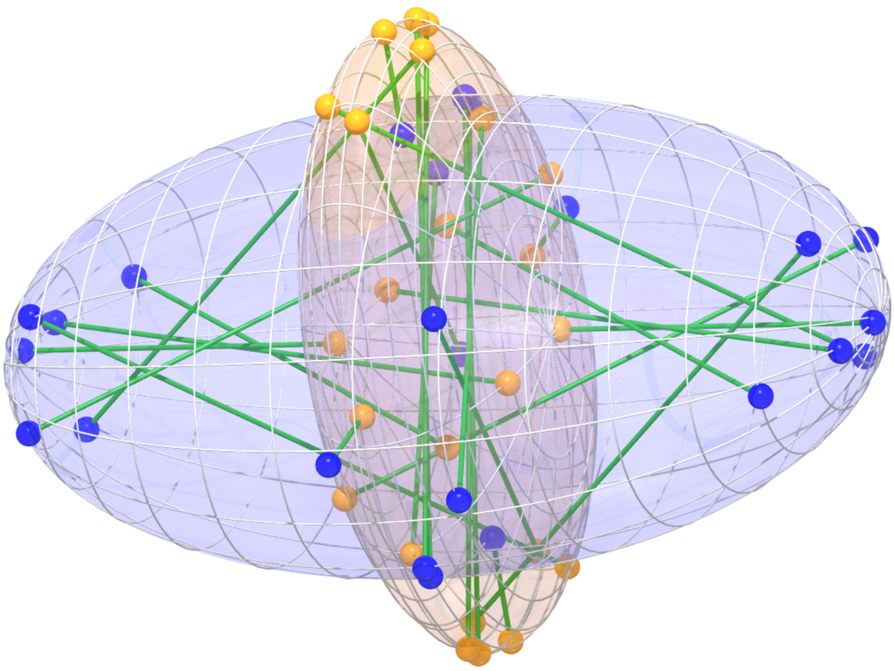

Figure 1: Two ellipsoids with 24 common normals

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Is it a Cube?

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Cube is one of the most fundamental spatial shapes we can draw and can observe from a drawing. But does (and if yes, when) a specific axonometric or perspective image really depict a cube and not a general cuboid? And can we observe this difference? In this presentation, we look for an answer to whether there is a common visual sense in this observation, whether there is a specific ratio of width, height and depth of the figure along the three dimensions when people generally feel that they are really watching a cube (see, e.g., Figures 1 and 2).

To do this, we conducted an experiment with an interactive computer model in which the width and height of the cuboid drawing are fixed, but the foreshortening in the third direction (the "depth") can be changed by the user with simple means [1]. We were interested in how coherently people, who were actually first year students of art and engineering fields, judge this situation, to what extent the set foreshortenings coincide or differ for different students. In the case of perspective images we have also examined how these positions are coherent to the exact solution of this problem in the geometric sense [2, 3, 4].

Key words: axonometric drawing, perspective drawing, common understanding of drawings

MSC 2010: 97C30, 51N05

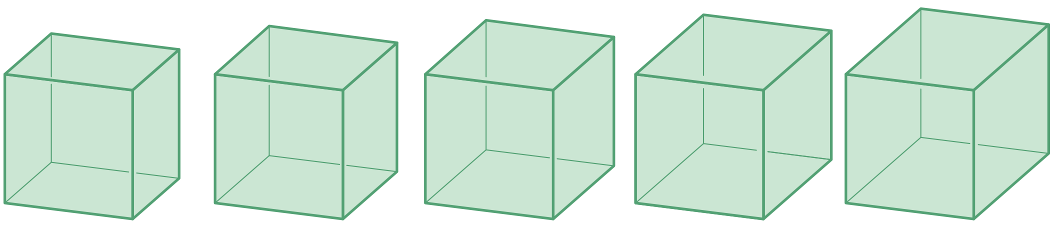

Figure 1: Five different axonometric images of a cuboid from the same direction of view, with fixed edge lengths in two directions ("height" and "width"), but an alternated foreshortening in the third direction ("depth") — Which one does resemble best a cube?

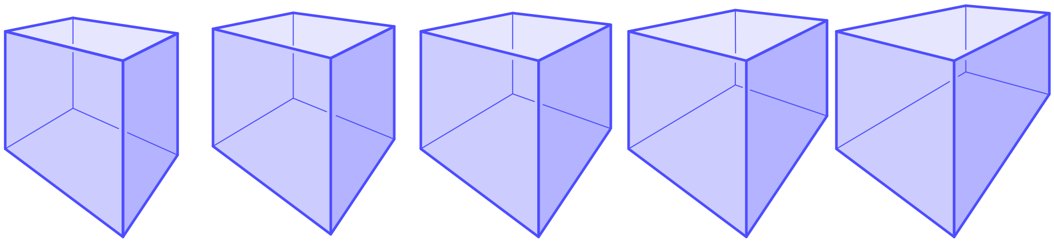

Figure 2: Five different perspective images of a cuboid from the same direction of view, with fixed edge lengths in two directions ("height" and "width"), but an alternated foreshortening in the third direction ("depth") — Which one does resemble best a cube?

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Möbius Linkages

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Möbius linkages are n-axes closed kinematic chains where the oriented axes form a discrete Möbius strip (Fig.1). These types of linkages have been the topic of two disputed papers [1, 2]. In both papers some surprising kinematic properties of these linkages are presented. These properties concern most of all their strange mobility. Unfortunately for all of the conjectured properties only numerical evidences were presented.

In the presentation the unexpected degree of freedom for the 8R Möbius linkage is proven and put into context with the 7R Möbius linkage. The proof also unveils some interesting geometric features of these linkages.

Key words: closed linkage, Möbius strip, strange mobility

MSC 2010: 70B15

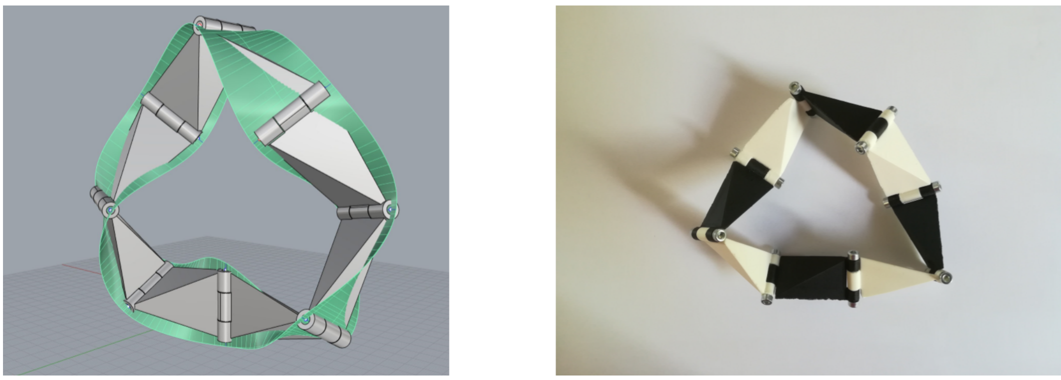

Figure 1: 8R Möbius linkage

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Curves of Brocard Points in Triangle Pencils in Isotropic Plane

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In this presentation we consider a triangle pencil in an isotropic plane consisting of the triangles that have the same circumscribed circle and study the locus of their Brocard points.

The *first Brocard point* P_1 of a triangle ABC is defined as the point such that its connection lines with the vertices A, B, C form equal angles with the sides AB, BC, and CA, respectively. In a similar manner, the *second Brocard point* P_2 is defined as the point such that the lines P_2A, P_2B and P_2C form equal angles with the sides AC, CB and BA, respectively.

In order to get the pencil of the triangles, we keep the vertices A and B fixed and move vertex C along the given circumscribed circle k. The curve of the first Brocard points of all triangles ABC is a curve of order 4. It has a cusp in the point A and touches k at the point B. An analog claim holds for the curve of the second Brocard points, Figure 1.

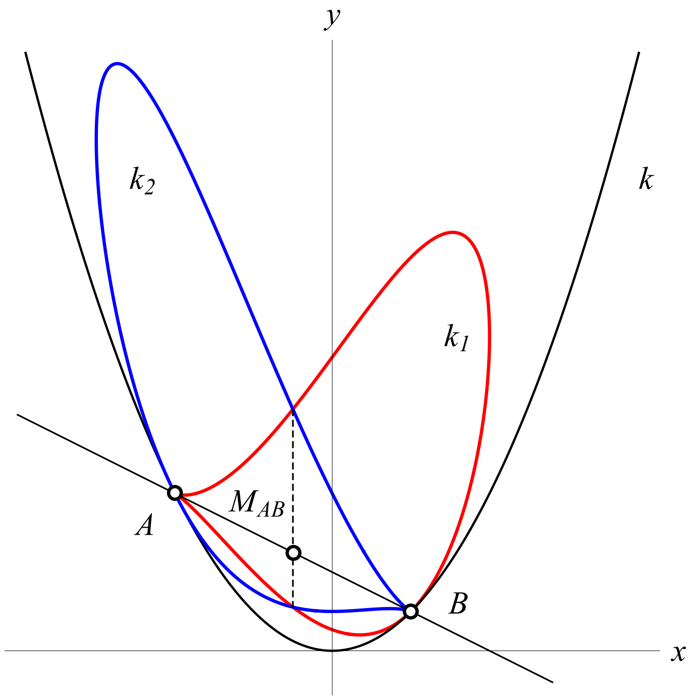

Figure 1: The curves k_1 and k_2 of the first and second Brocard points for a pencil of triangles with the same circumscribed circle k

In the second part of the presentation, we study the two Brocard curves of the pencil of tangential triangles. Keeping the points A and B fixed and moving C on the circle k, the pencil of tangential triangles $A_tB_tC_t$ that have the same inscribed circle k is obtained. The curves of their Brocard points are the curves of order 5, Figure 2.

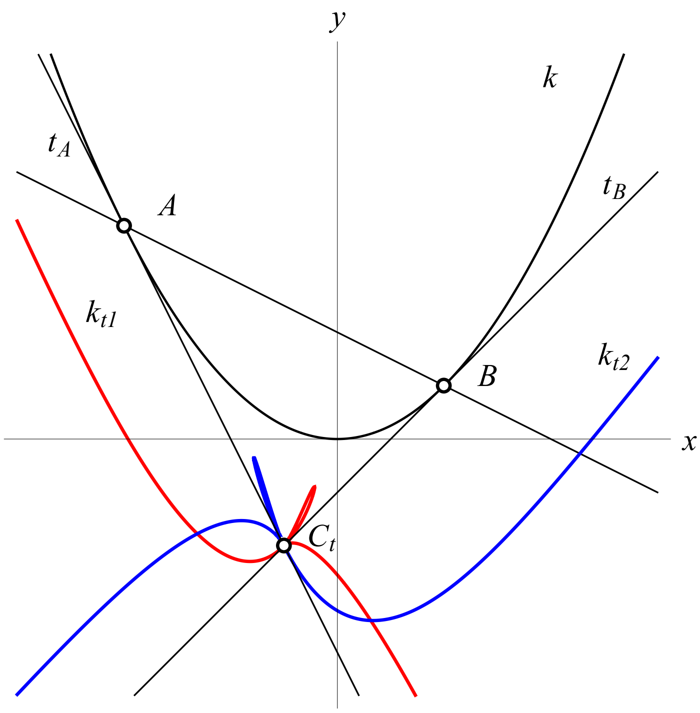

Figure 2: The Brocard curves k_{t1} and k_{t2} in the pencil of tangential triangles of the triangles with the same circumscribed circle k

Key words: isotropic plane, Brocard points

MSC 2010: 51N25

Quasi-Hyperbolic Plane $\mathfrak{G}(\mathbb{QH}_2)$ via Geometric Algebra

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Firstly, we present a new and not fully employed geometric algebra model that can serve as a basis for treating geometrical objects within the Cayley-Klein projective-metric planes based on the Gunn's projective geometric approach, [1]. However, our approach mainly differs in the geometrical interpretation of the algebraic objects where we incorporate G. K. C. von Staudt's point of view in algebraization of the geometrical structures based on the preservation of the harmonic relation among object-quadruples, [3]–[6]. The basic geometric objects within this model in linear, i.e. one-dimensional, and planar, i.e. two-dimensional case, are discussed in detail before we turn to a discussion of some metric-specific features.

In the second part, we introduce the hyperbolic measure of distance and parabolic measure of angle into our projective geometric algebra model in order to obtain the *quasi-hyperbolic* plane $\mathfrak{G}(\mathbb{QH}_2)$, one of nine Cayley-Klein's projective-metric planes, [2]. Also, we define in $\mathfrak{G}(\mathbb{QH}_2)$ a non-involutorial birational quadratic mapping named *automorphic Maclaurin mapping* and we analyze some of its projective and affine properties in relation to the circular curve construction problem in the quasi-hyperbolic plane.

Key words: projective geometric algebra, quasy-hyperbolic plane, quadratic transformation

MSC 2010: 51A05, 51M99, 51N15

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Surgeries of Gieseking Manifold via Computer Figures in $\mathbf{C}_\infty$ for Hyperbolic Space $\mathbf{H}^3$

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To Memory of Professor Gyula (Julius) Strommer on the 101st Anniversary of His Birth

In our Novi Sad conference paper (1999, [1]) we described Dehn type surgeries of the famous Gieseking (1912) hyperbolic ideal simplex manifold S, leading to compact fundamental domain $S(k)$, $k = 2, 3, ...$ with singularity geodesics of rotation order k, but as later turned out with cone angle $2\pi(k-1)/k$. We computed also the volume of $S(k)$, tending to zero if k goes to infinity.

That time we naively thought that we obtained orbifolds with the above surprising property. As the reviewer of Math. Rev., Kevin P. Scannell (MR1770996 (2001g:57030)) rightly remarked, "this is in conflict with the well known theorem of D. A. Kazhdan and G. A. Margulis (1968) and with the work of Thurston, describing the geometric convergence of orbifolds under large Dehn fillings".

In our paper [2] we have updated our previous publication [2]. Correctly, we have obtained cone manifolds (for $k > 2$), as A. D. Mednykh and V. S. Petrov (2006) kindly pointed out. We completed our discussion and derived the above cone manifold series (Gies.1 and Gies.2) - by computer figures as well - in two geometrically equivalent form, by the half turn symmetry of any ideal simplex. Moreover, we obtained a second orbifold series (Gies.3 and 4), tending to the regular ideal simplex as the original Gieseking manifold.

We have already extended this method onto the 3 ideal double-simplex manifolds in our newer publication [3], being indicated in this presentation as well.

Key words: hyperbolic manifold by fundamental polyhedron, Gieseking manifold, Dehn surgeries, volume by Lobachevsky function

MSC 2010: 57M50, 57N10

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A Rarity in Geometry: a Septic Curve

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We study the locus $\mathcal{C}$ of all points in the plane whose feet of the normals to the six sides of a complete quadrilateral lie on a conic. In the Euclidean plane, it turns out that $\mathcal{C}$ is an algebraic curve of degree 7 and genus 5 and not of degree 12 as it could be expected. Septic curves occur rather seldom in geometry: Only 13 special curves of degree 7 are of particular meaning and 12 of them are mentioned on B. Gibert's page [3]. A rational septic related to the triangle appears in [4]. This motivates a detailed study of this particular curve (cf. Figure 1).

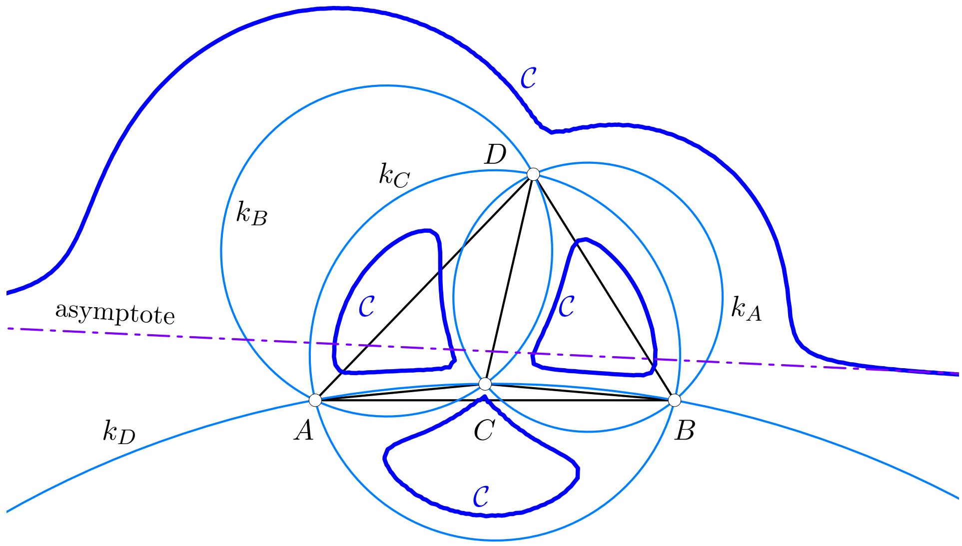

Figure 1: The septic locus $\mathcal{C}$ of points with conconic feet on the sides of a complete quadrilateral $\mathcal{Q} = ABCD$.

We determine its singularities and focal points. Then, we shall determine those points on $\mathcal{C}$ whose pedal conics degenerate which is the case for the three Miquel points of $\mathcal{Q}$ and four real further poins.

The curve $\mathcal{C}$ also shows up as the answer to a more general question including the locus of points whose reflections in $\mathcal{Q}$'s side lines are conconic (cf. Figure 2).

Finally, the possible degeneracy of $\mathcal{C}$ is payed attention to depending on the shape of the initial quadrilateral.

Some results are obtained by brute force computations while others can be deduced in a purely synthetic way.

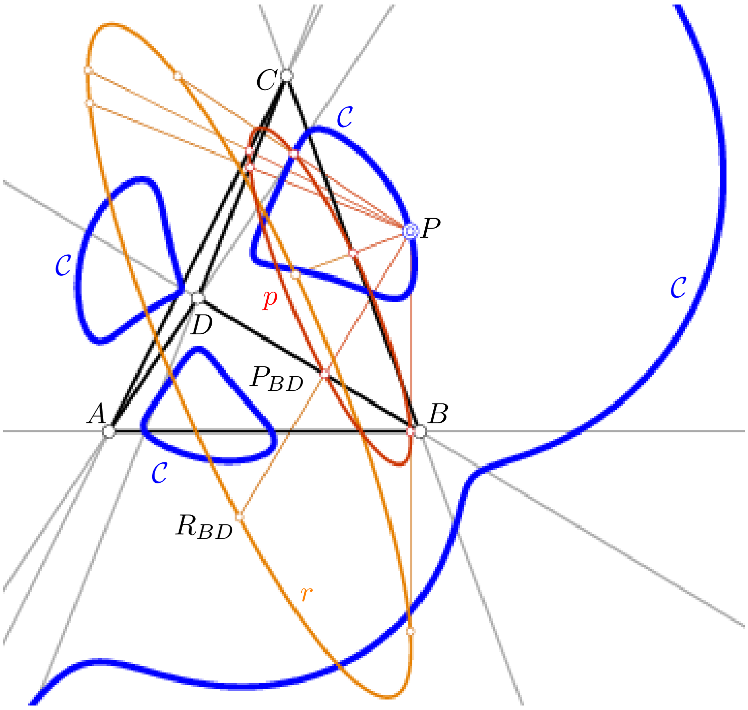

Figure 2: The conics p and r collect the pedal points and reflections of $X \in \mathcal{C}$.

Key words: quadrilateral, complete quadrilateral, pedal point, conic, six conconic points, septic curve, Simson line, Miquel point

MSC 2020: 14H45, 14P99, 51F99, 51N15

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Null Scrolls with Prescribed Curvatures in Lorentz-Minkowski 3-space

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In Lorentz-Minkowski 3-space, null scrolls are timelike ruled surfaces whose base curve $c(u)$ and rulings $e(u)$ are null (lightlike). In present work, we analyse the problem of finding null scrolls with prescribed mean, as well as Gaussian curvature. By introducing the special frame $L(u)$ of the base curve $c(u)$, related to the rulings of a null scroll, we can relate the curvature of a null scroll with the curvature of its base curve. The relation is given by the following first-order differential equation

$$2(k_1' - H + k_L) + k_1^2 = 0,$$

where k_L is the lightlike curvature of the base curve, H is the mean curvature of the null scroll and k_1 is the curvature of the base curve with respect to the frame $L(u)$. Conditioned by this equation, using the relations between frame related to the curvature k_L and frame $L(u)$, we can determine the family of null scrolls with a given null base curve and prescribed curvature.

Key words: null scroll, prescribed curvature, Lorentz-Minkowski space

MSC 2010: 53A10, 53C50

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Affine Regularization of Planar n-gons in a Finite Number of Steps

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We start with a generic planar n-gon Q_0 with vertices $q_{j,0}$ $(j = 0, \dots, n-1)$ and a given set of real numbers $u_k, v_k, w_k \in \mathbb{R}$ with $u_k + v_k + w_k = 1$ for $k \in K := \{1, \dots, k^* := \lfloor n/2 \rfloor - 1\}$. We iteratively define n-gons Q_k of generation $k \in K$ with vertices $q_{j,k}$ $(j = 0, \dots, n-1)$ via the barycentric combination $q_{j,k} := u_k\, q_{j,k-1} + v_k\, q_{j+1,k-1} + w_k\, q_{j+2,k-1}$. This defines a series of affine constructions of the vertices $q_{j,k}$ of Q_k from those of the predecessor polygon Q_{k-1}. In former papers we applied one iterative procedure to a given starting polygon; in a way, the arising infinite series of polygons gradually got 'regularized' (see [4] and [2]). Here we use only a finite number of steps. In order to achieve the regularization in this case we have to apply different appropriate affine constructions in every iteration step. The overall number of steps depends on the number n of vertices of Q_0.

We focus on the following

Statement: *There exist $u_k, v_k, w_k \in \mathbb{R}$ with $u_k + v_k + w_k = 1$ for all $k \in K$ such that this finite affine iteration process for general input data of Q_0 regularizes the polygon in the following affine sense: The final generation polygon Q_{k^*} is affinely equivalent to a given regular prototype n-gon Π.*

In this paper we prove this very surprising statement. The proof will make use of the fact that any planar n-gon Q_k can be interpreted as a vector $(q_{0,k}, \dots, q_{n-1,k})^t \in \mathbb{C}^n$. The described affine constructions of generation Q_k from the predecessor Q_{k-1} then can be described by $n \times n$ matrices $M_k \in \mathbb{R}_{n \times n}$. These matrices M_k depend on (u_k, v_k, w_k) and turn out to be circulant. Well-known results on the Eigenvectors and Eigenvalues of such matrices contribute to our proof. They enable us to determine and characterize the desired series of reals u_k, v_k, w_k such that at each step from Q_{k-1} to Q_k two (in special cases: one) of these Eigenvalues vanish.

Remarks:

- This result generalizes results by P. Pech [3], F. Schmidt [5] and I.J. Schoenberg [6] for a finite step regularization in a Euclidean setting.
- Our constructions deliver affinely regular n-gons Q_{k^*}. According to A. Barlotti [1] these affinely regular n-gons Q_{k^*} could be regularized (in a Euclidean setting) by one further (Euclidean) construction step.
- Fig. 1 displays a typical example for the case of pentagons $(n = 5)$. As we have $k^* = 1$ here there exist affine regularizations in one single step. There exist triples $(u_1, v_1, w_1)^t \in \mathbb{R}^3$ such that the next generation pentagon is an affine image of a regular pentagonal star or of a regular pentagon. Q_1 is generated with $u_1 = w_1 \approx 0.723607, v_1 \approx -0.447214$ while Q_1^* is gained with $u_1^* = w_1^* \approx 0.276393, v_1^* \approx 0.447214$. Both possibilities are displayed in Fig. 1.

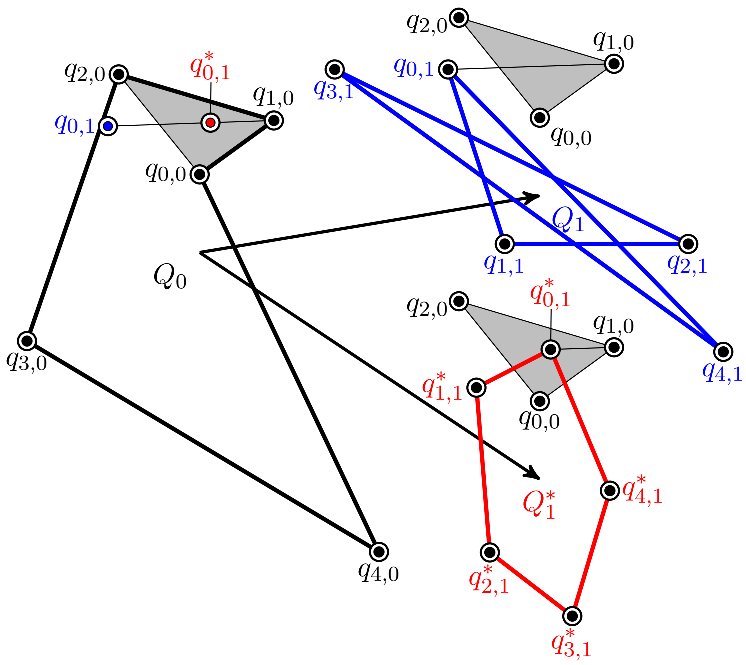

Figure 1: Affine regularization of the pentagon Q_0 (vertices $q_{0,0}, \ldots, q_{4,0}$) in one step (results are shifted by the indicated vectors): Q_1 (blue – vertices $q_{0,1}, \ldots, q_{4,1}$) is an affine regular pentagonal star while Q^*_1 (red – vertices $q^*_{0,1}, \ldots, q^*_{4,1}$) is an affine regular pentagon.

Key words: Finite Affine Iterations, Affine Regularization, Regular n-gons

MSC 2010: 51N10

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Geometry of Split Quaternion Factorization II

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Factorizations of split quaternion polynomials into linear polynomials can be computed by means of an algorithm developed from the factorization theory for polynomials over Hamiltonian quaternions [2]. Generically, the algorithm will succeed, but due to the presence of zero divisors it fails in special cases. Even factorization of quadratic split quaternion polynomials is not trivial [1, 4].

We consider polynomials P such that the rational curve parametrized by P in the projective space of split quaternions is not contained in the quadric of zero divisors. In order to compute factorizations of polynomials in this class the factorization algorithm can be adapted by investigation on its "geometry". Provided a factorization into linear polynomials exists, the new algorithm will find it and thus yields a characterization of factorizability [3].

Factorization of split quaternion polynomials also provides us with a better understanding for some phenomena observed in the factorization theory for dual quaternion polynomials which is highly relevant in space kinematics [2]. Our improved algorithm can compute new factorizations of certain dual quaternion polynomials that are overlooked by existing algorithms.

Acknowledgement: Daniel F. Scharler was supported by the Austrian Science Fund (FWF): P 31061 (The Algebra of Motions in 3-Space).

Key words: skew polynomial ring, null quadric, left/right ruling, Clifford translation

MSC 2010: 12D05, 51N15, 51N25

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Billiards in Ellipses and Ellipsoids

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A *billiard* is the trajectory of a mass point in a domain with ideal physical reflections in the boundary. Already for two centuries, billiards in ellipses have attracted the attention of mathematicians, beginning with J.-V. Poncelet and C.G.J. Jacobi. In 2005 S. Tabachnikov published a book on billiards as integrable systems [3]. Recent computer animations of billiards in ellipses, which were carried out by D. Reznik [1], stimulated a new vivid interest on this well studied topic, where algebraic and analytic methods are meeting.

It is wellknown that the sides of a billiard in an ellipse e are tangent to a confocal conic c called *caustic* which can be an ellipse or hyperbola. We speak briefly of an *elliptic* or *hyperbolic* billiard. If one billiard inscribed in e and tangent to c closes after N reflections, then it is called *periodic* and closes for each choice of the initial vertex. When this vertex varies on e, then this defines a socalled *billiard motion* along e with fixed caustic c. This variation neither preserves angles or side lengths nor is a projective motion. However, among many other invariants the perimeter of the billiard and the sum of Cosines of the interior angles remain constant.

The goal of this presentation is to show that there is a one-to-one correspondence between elliptic and hyperbolic billiards which preserves the lengths of corresponding sides (Figure 1). There is even a continuous transition from one type to the other via isometric focal billiards in an ellipsoid. This transition can be used to transfer results concerning billiard motions and invariants from the plane to three dimensions. Moreover, there is a parametrization of focal billiards in ellipsoids in terms of Jacobian elliptic functions, which sends a square grid together with diagonals to the Poncelet grid of a focal billiard on a one-sheeted hyperboloid.

Key words: ellipse, ellipsoid, billiard, caustic, Poncelet grid, billiard motion, focal billiard, Jacobian elliptic functions

MSC 2010: 51N20, 53A05, 37D50, 33E05

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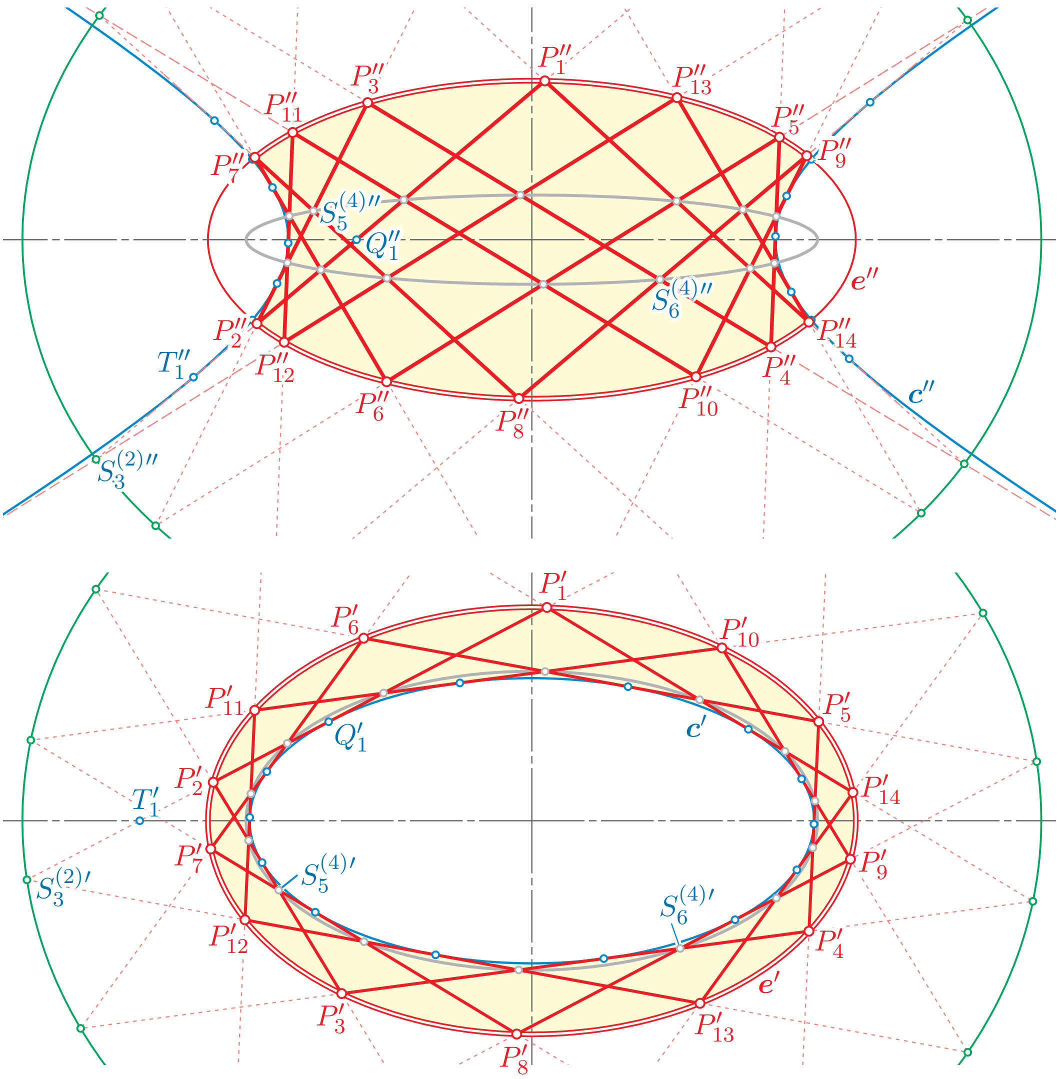

Figure 1: A hyperbolic and an elliptic periodic billiard with $N = 14$ and turning number $\tau = 3$. The two billiards are isometric, i.e., $\overline{P'_i P'_{i+1}} = \overline{P''_i P''_{i+1}}$ for all $i \in \{1, 2, \ldots, 14\}$.

Polyhedrons whose Faces are Special Quadric Patches

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We seize an idea of Oswald Giering [1], who replaced pairs of faces of a polyhedron by patches of hyperbolic paraboloids and link up edge-quadrilaterals of a polyhedron with the pencil of quadrics determined by that quadrilateral. Obviously only ruled quadrics can occur. There is a simple criterion for the existence of a ruled hyperboloid of revolution through an arbitrarily given quadrilateral. Especially, if a (not planar) quadrilateral allows one symmetry, there exists one such hyperboloid of revolution through it (Figure 1), and if the quadrilateral happens to be equilateral, the pencil of quadrics through it contains two hyperboloids of revolution with orthogonal axes. To mention an example, for right double pyramids the axes of the hyperboloids of revolution are, on one hand, located in the plane of the regular guiding polygon, and on the other, they are parallel to the symmetry axis of the double pyramid.

Not only for platonic solids, but for all polyhedrons, where one can define edge-quadrilaterals, their pairs of face-triangles can be replaced by quadric patches, and by this one could generate roofing of architectural relevance. Especially patches of hyperbolic paraboloids or, as we present here, patches of hyperboloids of revolution deliver versions of such roofing, which are also practically simple to realize.

Key words: polyhedron, quadric, hyperboloid of revolution, Bézier patch

MSC 2010: 51Mxx, (51M20, 51M30), 51N05, 51N20, 15Axx

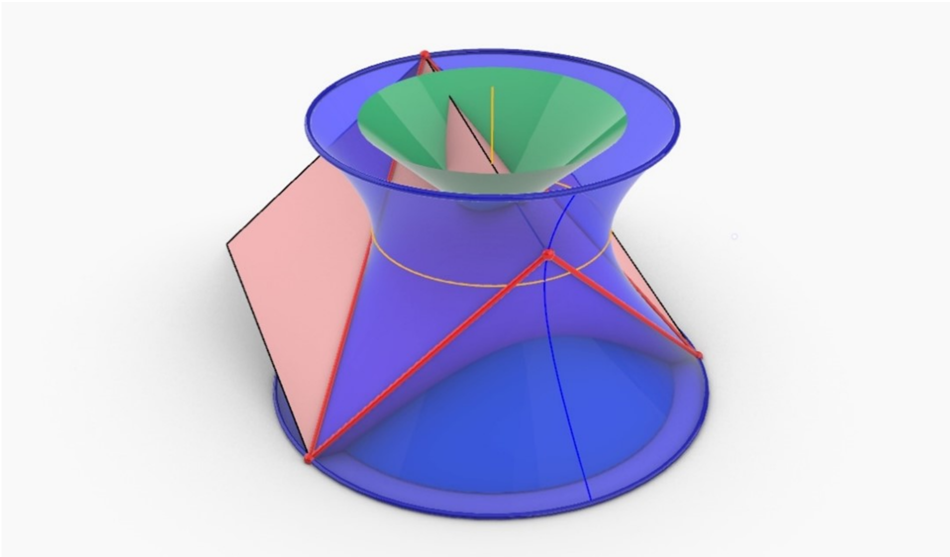

Figure 1: A hyperboloid of revolution through two edges and two face-diagonals of a cube

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On Generalized Hadwiger Numbers of a Convex Body

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A packing is a collection of convex bodies in $\mathbb{R}^d$ whose interiors are pairwise disjoint. Recall that two convex bodies touch each other if their intersection is nonempty but their interiors are disjoint. The contact graph of a packing is a graph whose vertices are in one-to-one correspondence with the convex bodies of the packing, and two vertices are connected by an edge exactly when their corresponding convex bodies are touching in the packing.

The Hadwiger number $H(K)$ of a convex body K is the maximum number of mutually nonoverlapping translates of K that can touch K. $H(K)$ is often called the translative kissing number of K as well. It is known that $d^2 + d \leq H(K) \leq 3^d - 1$ for every d-dimensional convex body K (cf. [1], [2]). For 3-dimensional convex bodies we have $12 \leq H(K) \leq 26$. In some cases, determining the exact value of $H(K)$ is difficult, but some other extremal properties of the contact graphs of those packings in which every member is a translate of K and it touches K can be found more easily. Let's say the maximum vertex degree can be such a property in those contact graphs.

We generalize the Hadwiger number $H(K)$ of a convex body K in the following way: The nth Hadwiger number $H_n(K)$ of a convex body K is the maximum number of mutually nonoverlapping translates of K that can touch each element of a collection of n pairwise touching translates of K. Note that $H_n(K)$ is defined exactly when there are n pairwise touching translates of K, so $1 \leq n \leq t(K)$, where $t(K)$ is the touching number of K, that is the maximum number m for which there exist m pairwise touching translates of K. Of course, $H_1(K) = H(K)$. $H_2(K)$ is the maximum vertex degree appearing in the contact graphs of those packings in which every member is a translate of K and it touches K.

We prove the inequality $H_2(K) \leq 2 \cdot 3^{d-1} - 2$ with equality only for parallelotopes, implying $H_2(K) \leq 16$ for 3-dimensional convex bodies. We conjecture $H_2(K) \geq 5$ and $H_3(K) \leq 9$ for any 3-dimensional convex body K, with equality in the upper bound only for parallelepipeds.

Key words: Hadwiger number, translative kissing number, touching number, translative packing, contact graph

MSC 2010: 52C17, 52A40

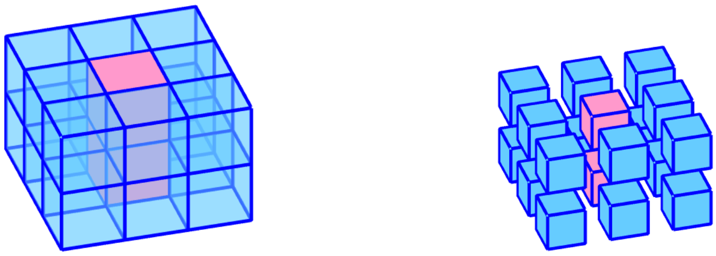

Figure 1: A packing of 18 unit cubes in the $3 \times 3 \times 2$ cubic grid showing $H_2(C) \geq 16$ for the 3-dimensional unit cube C, using 16 transparent blue cubes. Every blue cube touches both pink cubes. In the second part of the figure, the construction is repeated by using shrunken copies of the cubes (with no transparency) to achieve better visibility of the configuration structure.

Figure 2: A packing of 12 unit cubes in the $3 \times 2 \times 2$ cubic grid showing $H_3(C) \geq 9$ for the 3-dimensional unit cube C, using 9 transparent blue cubes. Every blue cube touches all three pink cubes. In the second part of the figure, the construction is repeated by using shrunken copies of the cubes (with no transparency) to achieve better visibility of the configuration structure.

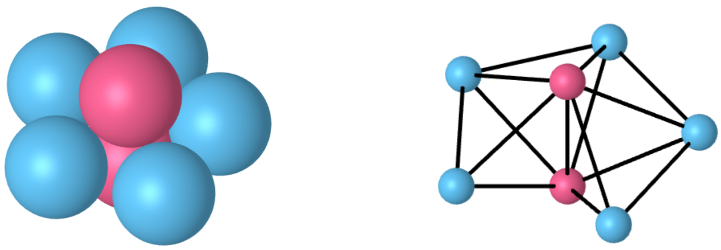

Figure 3: A packing of 7 unit spheres showing $H_2(S) \geq 5$ for the 3-dimensional unit sphere S. Every blue sphere touches both pink spheres. In the second part of the figure, the contact graph of this packing is displayed in which the vertices are shrunken copies of the spheres.

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Nonstandard Subdivision Masks

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In Computer Aided Design subdivision curves and surfaces are commonly used and well-studied. One of the most famous techniques is the Chaikin's algorithm [1, 2] and it is known that with the Lane-Riesenfeld algorithm the B-Spline curves with the degree of n can be defined [3]. These algorithms can be defined with subdivision masks. A subdivision mask is a series of real numbers used by the algorithms. In case of the Lane-Riesenfeld algorithm the mask is

$$r = \frac{1}{2^n}\left(\binom{n}{0}, \binom{n}{1}, \cdots, \binom{n}{n}\right).$$

To find and analyze new types of subdivision masks and algorithms is a current research area of computer graphics. Different kinds of solutions can be found in e.g. [4, 5]. In this presentation we describe new types of subdivision masks to define new curves with the algorithms above. These subdivision masks have at least one parameter besides the parameter n therefore the curves have one or two shape parameters. We present the necessary requirements of subdivision masks, which our masks meet and some of their geometric properties.

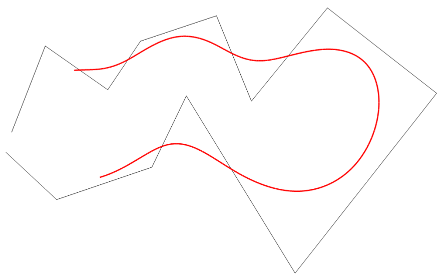

Figure 1: A trigonometric function based subdivision curve with diferent shape parameters

Key words: CAD, subdivision, freeform curve

MSC 2010: 65D17

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Equidistant

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The adjective "equidistant" originated from Late Latin word aequidistantem (*aequī* = equal + *distantem* = distant) and Middle French word équidistant. In general, "equidistant" means equally distant from one or more objects, or located at the same distance, occupying a position in equal distance between several objects. An equidistant set (also called a midset, or a bisector) is a set each of whose elements has the same distance (measured using some appropriate distance function) from two or more sets. The serious study of the properties of equidistant sets as mathematical objects was initiated quite recently, in 1970's, see [1], [2].

Set of points equidistant to each other depends on the space dimension (2 points on line, 3 points in plane, 4 points in space, etc). To find a locus of points equidistant to fixed geometric figure (line, curve, surface or discrete set of points) is a geometric construction technique frequently used in many applications. Median axis (or Bloom topological skeleton) of an object (whose closure is also reffered to as the cut locus) is example used in shape analysis.

Basic concept of an equidistant to a manifold reveals a bundle of problems connected to intrinsic geometric properties of resulting geometric figure. The study of equidistant sets is more interesting in the case when the background metric space is the non-Euclidean space. To find a curve equidistant from a given curve on a surface seems to be a quite interesting problem of classical differential geometry with left open questions. Some of these questions will be addressed, and possible approach will be presented on searching for answers.

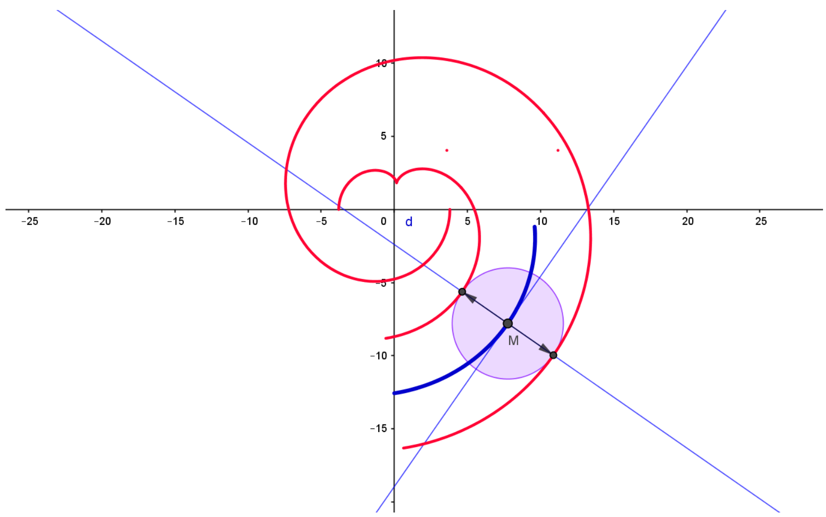

Figure 1: Equidistant curves to spiral

Key words: equidistant set, locus of points, equidistant curves on a surface

MSC 2010: 53C30, 51K10

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Chains of Hyperosculating Conics

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Two hyper-osculating conics span a pencil of conics with a single singular curve of 2^{nd} order, namely their single, twice counting common tangent. In the following the place of action is the projectively enclosed (real) affine plane, endowed with an affine and finally a Euclidean structure. There exist “spirals” formed by similar quarter-ellipses, whereby the main vertex of one arc is the second vertex of the adjacent arc. Such spirals form “infinite chains of hyper-osculating conic arcs”. Generalising such a chain to GC^3- splines with conic arcs leads to the problem of finding an intermediate conic hyper-osculating two conics, which belong to two given pencils of hyper-osculating conics, so-called “HO-pencils”. Another question asks for the shortest set of consecutively hyper-osculating intermediate conics connecting two given conics. As there will occur special cases it is necessary to provide criteria for two HO-elements belonging to one single conic.

A Survey of Structural Properties of Voronoi Diagrams

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Voronoi diagrams are a tessellation technique, which subdivides space into regions in proximity to a given set of objects called seeds. As a result of their aesthetically pleasing properties, they have been widely used in art and design.

Moreover, patterns emerging naturally in biological processes (for example, in cell tissue) can be modeled in a biomimicry process via Voronoi diagrams. As they originate in nature, we investigate the physical properties of such patterns to determine whether they are optimal given the constraints imposed by surrounding geometry and natural forces.

This paper describes under what circumstances the Voronoi tessellation has optimal (structural) properties by surveying recent studies that apply this tessellation technique across different scales. These applications range from construction of foam-like materials, 3D-printed hollowed-out objects, to facades of tall buildings. We find that some optimization methods are employed to reach a structure which satisfies defined constraints.

To investigate the properties of random Voronoi tessellations in comparison to other 3D tessellation methods, we additionally run and evaluate a simulation in Karamba3D, a parametric structural engineering tool for Rhinoceros3D.

Figure 1 shows foam from a water-soap solution after having rested overnight in a transparent cube. The bubbles form a tessellation akin to an optimised three-dimensional Voronoi diagram shown in Figure 2.

Keywords: Voronoi diagrams, 3D tessellations, 3D scaffolds

MSC 2010: 51-02, 52C25, 05B45

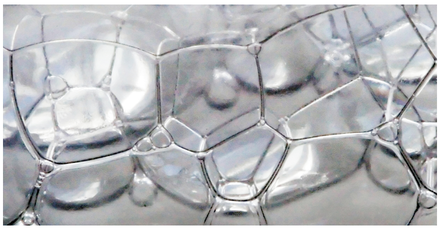

Figure 1: Foam within transparent cube

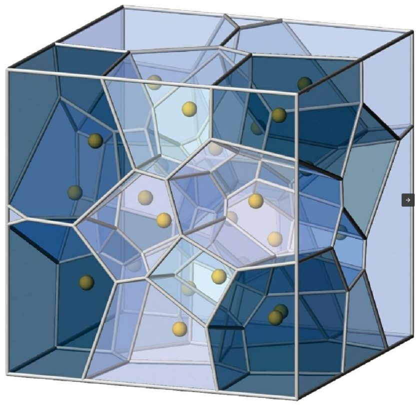

Figure 2: Consolidated Voronoi diagram in 3D

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Visualization of the Optimal Ball and Horoball Packings Related to Coxeter Tilings Generated by Simply Truncated Orthoschemes with Parallel Faces

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In this visualization, we consider the ball and horoball packings belonging to 3-dimensional Coxeter tilings that is derived by simply truncated orthoschemes with parallel faces. The goal of this investigation is to study the structure of the ball and horoball packing arrangements for all above Coxeter tilings in hyperbolic 3-space $\mathbb{H}^3$. The result of this visualization is useful for determining the optimal ball and horoball packing arrangements and their densities.

The Schläfli symbol of the considered Coxeter tilings is (∞, q, r, ∞), where $\frac{1}{q} + \frac{1}{r} \geq \frac{1}{2}$, see [1]. We use for the computations the Beltrami-Cayley-Klein sphere model where we construct the corresponding orthoschemes. Each of them has an ultra ideal vertex therefore we truncate it with its polar (truncation) hyperplane. The corresponding ball and horoball packings are generated by these truncated orthoschemes.

As indicated on the Schläfli symbol, it has also two pairs of parallel faces that meet each other at tangent lines of the model sphere, and their common point lies at the infinity, too.

We construct inballs (inscribed spheres) related to the truncated orthoschemes. Generally, in each case the inball will touch 4 of 5 faces therefore, in every case we have 5 candidates of densest inball packing arrangement, [3].

Finally, we visualize the horoball packings. The centers of horoballs are required to lie at vertices of the polyhedral cells constituting the tiling. We allow horoballs of different types at the various vertices. We find that in cases $(\infty, 3, 6, \infty)$, $(\infty, 4, 4, \infty)$, $(\infty, 6, 3, \infty)$, there are two ideal vertices thus we can place two horoballs altogether such that these touch each other on the edge connecting the two ideal vertices. The future interesting investigation is finding corresponding packing densities, using the approach based on [4, 5, 6]. In this talk we show some interesting very dense horoball configurations [7].

Key words: Coxeter group, horoball, hyperbolic geometry, packing, tiling

MSC 2010: 52C17, 52C22, 52B15

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Elliptical Cones, Development & Plumbing

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It is a far cry from underlying geometry of quadrics to a practical application of elliptical cones that intersect on a given circle and an ellipse so as to simplify the design of a piece of plumbing, a developed asymmetric Y-pipe that joins three axially parallel, coplanar ducts. This is described succinctly but completely in the five illustrations that follow.

Key words: elliptical cone, conic intersection, modeling, development

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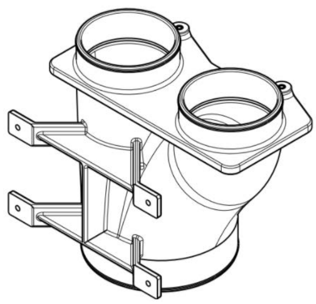

Figure 1: CAE Flight Simulator Cooling Air System Component

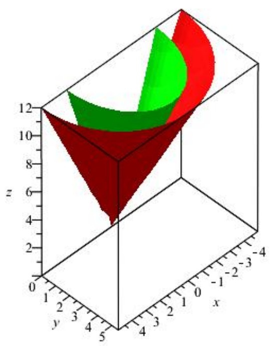

Figure 2: Two Standard Form Elliptical Cones

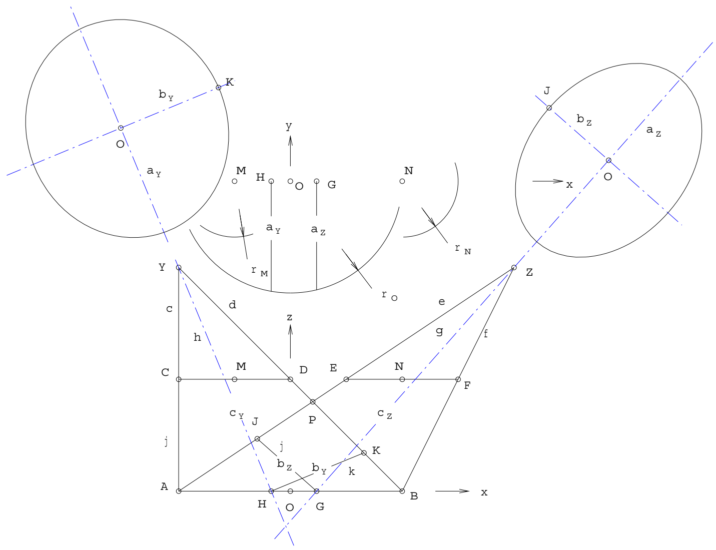

Figure 3: Solid Modeling Problem

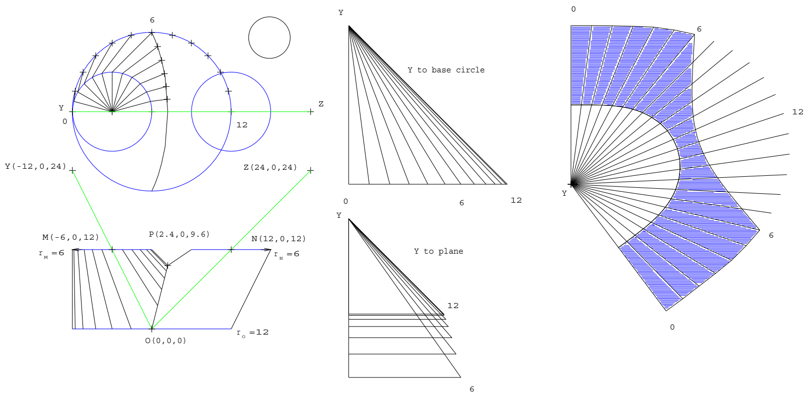

Figure 4: Development of the Cone on the Left

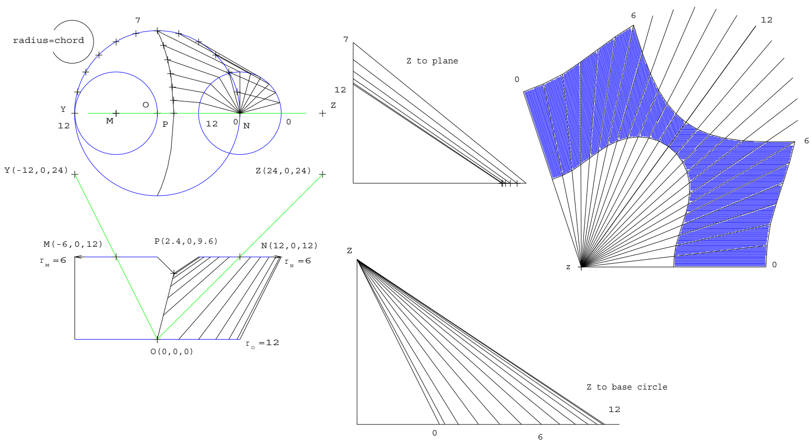

Figure 5: Development of the Cone on the Right

Posters

On Killing Magnetic Curves in Hyperboloid Model of $SL(2,\mathbb{R})$ Geometry

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Magnetic curves represent trajectories of charged particles moving on a Riemannian manifold under the action of a magnetic field. The study of magnetic curves in arbitrary Riemannian manifolds was developed in the early 1990's. Interesting results on magnetic curves in Euclidean space [5] and Sol space [2] appeared recently.

A vector field X is a *Killing vector field* if the Lie derivative with respect to X of the ambient space metric g vanishes. The Killing vector field can be interpreted as an infinitesimal generator of isometry on the manifold in the sense that the flow generated by this field is a continuous isometry of the manifold.

The trajectories corresponding to the Killing magnetic fields are called the *Killing magnetic curves*. Killing magnetic curves in Euclidean space $\mathbb{E}^3$, Sol space and $\mathbb{S}^2\times\mathbb{R}$ space were studied in [1, 3, 4] respectively.

We consider the Killing magnetic curves in hyperboloid model of $SL(2,R)$ space.

Key words: Magnetic curve, Killing vector field, SL$(2,\mathbb{R})$ geometry

MSC 2010: 53C30, 53C80

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Acrylic Adventures

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Geometry is art – shape, structure, form, relations, quantities, measures, order, regularity, rules and exceptions.

Art is craft – revival of the abstract. From ideas to material objects. Art is an expression of our knowledge, understanding and awareness.

In this poster I present my recent acrylic paintings.

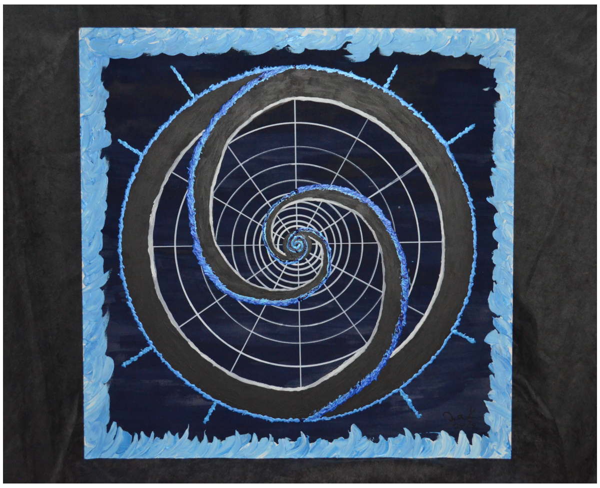

Figure 1: *Spiraling Sea*, acrylic on HDF board

Figure 2: *5 triangles*, acrylic on canvas

Hyperbolic Paraboloid in four "Perspectives"

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In this poster we will give an overview of a hyperbolic paraboloid. It will be presented in four different views: through differential geometry, student's view, an engineering standpoint and through representation as an anaglyph/perspective representation in 2-D. For the student's view we will use student's work from the elective course "Perspective" at the Faculty of Civil Engineering, University of Zagreb.

Key words: hyperbolic parabolid, ruled surface, perspective, applied geometry

MSC 2010: 65D17, 51N20, 53A15, 51N05

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