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ABSTRACTS

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Plenary lectures

The Mathematical Model of a Dome and Some Related Problems

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We investigate the mathematical model of a roof under ideal conditions, generalizing the catenary curve of the one-dimensional case. Under ideal conditions, the shape of a roof is a surface with the lowest center of gravity among all surfaces with the same area and the same boundary curve. These surfaces are called singular minimal surfaces. Among the examples, we focus on the axisymmetric surfaces showing their main properties. Finally, we give a comparison analysis between the mathematical model of a rotational dome (rotational tectum) and two known surfaces such as the catenary rotation surface and the paraboloid.

Key words: catenary, rotational tectum, paraboloid, center of gravity

MSC 2020: 53Z30, 53A10, 68N30

On Lightlike Curves and Surfaces in Lorentz-Minkowski Space

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We consider various aspects of lightlike curves and surfaces in Lorentz-Minkowski spaces of dimensions 3 and 4. First we analyse intrinsic geometry of curves lying in lightlike (hyper)planes and relate theories of curves in Lorentz-Minkowski spaces and in simply isotropic spaces by developing Darboux frames of curves in lightlike (hyper)planes. Based on these relationships, we characterize some special classes of curves lying in lightlike (hyper)planes. Next we present a Weierstrass-type representation formula for regular lightlike surfaces. It allows a local parametrization of a regular lightlike surface by a pair of holomorphic and meromorphic dual functions. We also analyse connections of this theory to the theory of surfaces in simply isotropic spaces. Finally, we bring up some educational aspects of teaching curves and surfaces in university undergraduate context.

Key words: lightlike curve, lightlike surface, isotropic space, mean curvature, Weierstrass representation formula

MSC 2020: 53A35, 53B30

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Construction of Plane Models of Modular Curves

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Joint work with Iva Kodrnja

We consider congruence subgroups $\Gamma_0(N)$, $N \geq 1$, and the corresponding compact Riemann surface $X_0(N)$ considered as a complex irreducible smooth projective curve. Let $\mathbb{H}$ be the upper half plane $\mathrm{Im}(z) > 0$. The group $\Gamma_0(N)$ acts on $\mathbb{H}$ by the Möbius transformations and the quotient $\Gamma_0(N)\backslash\mathbb{H}$ is an affine irreducible complex curve. Adding a finite set of points we obtain $X_0(N)$. If we take that f, g, h are linearly independent modular forms of same even weight $m \geq 2$ for $\Gamma_0(N)$, we can construct a regular map $X_0(N) \longrightarrow \mathbb{P}^2$ (a complex projective plane) initially defined on the affine curve $\Gamma_0(N)\backslash\mathbb{H}$ by $z \longmapsto (f(z) : g(z) : h(z))$. The image $\mathcal{C}(f, g, h)$ is an irreducible projective plane curve. Using SAGE software system we describe reduced equation of $\mathcal{C}(f, g, h)$, i.e., we compute irreducible homogeneous polynomial P such that $P(f(z), g(z), h(z)) = 0$. We will give a lot of concrete examples. We also discuss the question when $\mathcal{C}(f, g, h)$ is a model of $X_0(N)$.

The author acknowledges Croatian Science Foundation grant no. 3628.

Key words: modular forms, modular curves, birational equivalence

MSC 2020: 11E70, 22E50

Poncelet Porisms

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In this talk, we shall understand a porism as a geometric figure that closes independent of initial conditions. Among the many porisms, Chapple's porism, *i.e.*, the one-parameter family of triangles with a common circumcircle and a common incircle is, probably, best known. However, it is only a very special kind of porism among a huge variety of porisms.

We pay our attention to Poncelet porisms that deal with one-parameter families of n-gons interscribed between two or more conics. In fact, the most general form of Poncelet's theorem states (cf. [3]):
If an n-gon with vertices $P_1, \ldots, P_n$ on a conic c_0 whose sides(line) $[P_i, P_{i+1}]$ are tangent to a conic c_i (with $i \in \{1, \ldots, n\}$) closes, i.e.*, $P_{n+1} = P_1$ for one particular choice of $P_1 \in c_0$, then it closes for any choice of $P_1 \in c_0$, provided that the conics $c_0, c_1, \ldots, c_n$ belong to one pencil of conics.*

Chapple's porism clearly meets the requirements of Poncelet's theorem. Moreover, it can be generalized in several ways: The two circles span a hyperbolic pencil of circles, and therefore, it is nearby to ask for more general porisms in hyperbolic, elliptic, or even parabolic pencils of cirlces. Starting with the incircle and circumcircle of a triangle, we already have a poristic family of triangles. On the contrary, two circles with radii $R > r$ and central distance d define a poristic family if, and only if, $d^2 = R^2 - 2Rr$, *i.e.*, only if Euler's triangle formula is fulfilled (cf. [6]). We call this a closure condition. Many of these conditions for bicentric n-gons are known (cf. [5]) and can easily be determined with Cayley's formula (cf. [4]). The latter formula does not apply to poristic figures with more than two conics. So, for each new type, we have to find its closure condition.

Among other things, the presentation will show how to find such closure conditions (see cf. [2]). We will also compare earlier results on the poristic traces of triangle centers with the centers' traces in more general porisms (cf. [7, 8]). For example, triangle centers that trace a circle in the course of one round in Chapple's porism, trace curves of degree six or more in the more general forms of porisms (see [2, 1]). Especially, incenters and excenters (not really centers in the strict sense, see [6]) show a completely different behaviour. While they trace circles in Chapple's porism, their orbits in more general porisms spread and the difference between incenter and excenters tends to fade.

Key words: porism, Poncelet porism, closure condition, pencil of conics.

MSC 2020: 51M04, 51D30

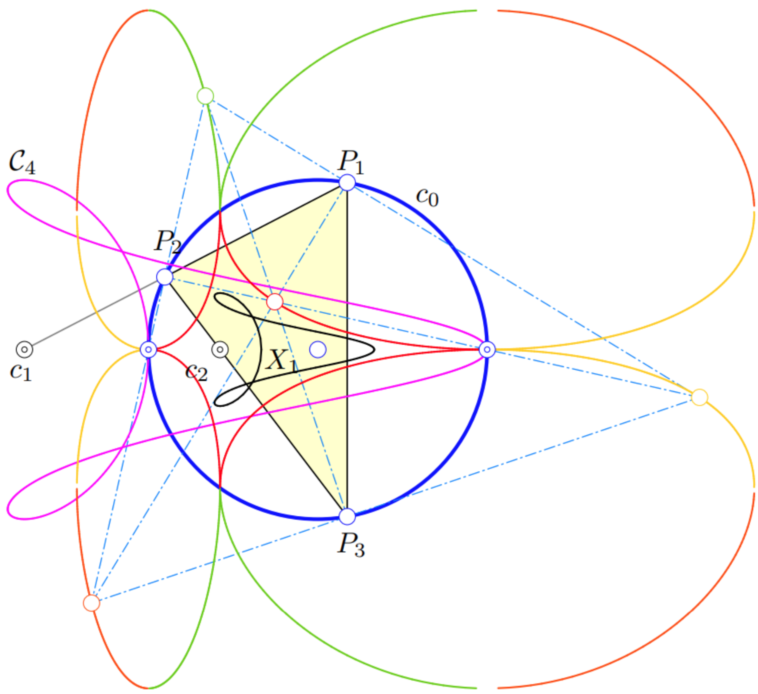

Figure 1: Poristic triangles in a hyperbolic pencil of circles (circumcircle c_0, and c_1, c_2 are the pencil's zero circles):
Parts of trace of the incenter X_1 belong to the trace of some excenter (algebraically), while the centroid X_2 (black trace) / or the orthocenter X_4 (magenta trace) does not have to share its orbit.

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Contributed talks

Teaching Descriptive Geometry in mathSTEM Context

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It was approximately at the same time when lockdown in 2020 started that Erasmus + project *mathSTEM: Teaching mathematics in STEM context for STEM students* was initiated by five universities from Croatia, North Macedonia and Turkey. Although its intellectual outputs were delayed by the complexity of the Covid situation, this project at the same time gained a new dimension due to this new moment in the world. Here we present how the concept of teaching descriptive geometry fit in the wider frame of mathSTEM teaching and present the contents of the e-platform that was hereby developed. We also give a short overview of LTT events in mathSTEM that were organized for students and for teachers.

Key words: STEM, descriptive geometry, geometry education

MSC 2020: 97G80

Mathe Schatzkiste: Interactive Multimodal Learning Environments in Primary Mathematics Education

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One of the most effective ways to teach mathematics in primary schools are hands-on activities. Students need to explore and to experience the teaching content with materials and well-constructed tasks in order to develop mathematical competences. This is of particular importance in the development of number concepts and spatial ability.

We started a research project to invent interactive multimodal learning environments based on the new curriculum, which is centered on four central subject concepts. Our goal is to create demanding and empirically proven tasks for each concept.

We will give a short overview of our project and materials so far and an outlook on our future plans.

Key words: mathematics education, learning environments, hands-on, spatial ability, number concepts, cognitive development

MSC 2020: 97Dxx

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On Basic Geometry and Graphics Literacy for Architects An Educational Trial

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Literacy is a permanent topic for teachers in general, especially in the era of IT, to which, as we know, the field of Geometry and Graphics is particularly sensitive [1]. After long academic hesitations, it became clear that the digital was neither in opposition nor alternative to Geometry and Graphics education. Instead, it could be regarded, at the same time, as an operational tool, a communication channel, and a disciplinary archive. At the point, software programs may be acknowledged as an interactive evolution of the traditional textbooks, with the additional benefit of the *interoperability*, which offers new bases for *interdisciplinarity* [2].

This was already clear relatively long time ago in the mind of the authors of the *Unesco / UIA Charter for Architectural Education (Tokyo 2011)*, often mentioned in my work, which recommends extending the digital turn to all the disciplinary fields involved in architectural education, be it science-, art-, or humanities-related. This was also a topic during the celebrations of the 150th anniversary of the Politecnico di Milano in 2013 [3]. After the event, a committee was designated to formulate new educational programs based on digital geometry and graphics. A so-called *Digiskills* program for the master courses was implemented, which is still part of our manifesto. It was addressed to advanced modeling, parametric modeling, and BIM. That is, quite far from the basic literacy level, yet.

In 2019, as the deputy coordinator of the Bachelor in Architectural Design, I was asked by my Dean and School Board to propose a new *Digiskills* program, finally addressed to the students of the second year. That is, to a properly said basic literacy level [4]. Taking inspiration from the Unesco/UIA recommendations, it was clear that the program was supposed to offer a *hub* of operational skills, as the base for an interdisciplinary approach to the digital, as it is required for architectural education. It was then a matter of how to integrate the two-component educational system based on *Geometry & Graphics* into a wider three-component system based on *Geometry & Graphics & Information* [5].

Keeping 3D space modeling, elaboration, and configuration firmly at the center of the program, as well as the basic connections with parametric and BIM environments, it was decided to make the program more adherent to the typical routines of the architectural design processes. According to this idea, hints concerning other relevant aspects normally orbiting around and feeding digital modeling in architecture were included, namely and mainly related to data *acquisition* (input) and *prototyping* (output), where the consubstantiality between Information and Geometry & Graphics would have emerged [6].

To meet the educational target in the short time of 4 ECTS, the 20 classes and the about 1.000 students involved, worked in parallel and synchronously week by week, in *distance-learning* mode, with pre-arranged ready-made digital materials, to be further edited and developed during lectures and homework. The results are yearly published on the official website of the Bachelor's program [7]. From the next year 2022/2023, a series of video-tutorial will also be provided, taking further advantage of the *flipped* approach.

At the very beginning of the semester, a geometry training on quadrics and NURBS in a purely 3D CAD environment is scheduled. In the following part, students recognize and manage geometry and graphics on the assigned architectural project and the surrounding urban area, through about ten selected software programs and various file formats, to be integrated at the end as a unitary digital model. On the way on, sometimes unexpectedly, they understand and experience the essential role played by geometry and graphics in all the software environments used, as well as in all the stages of the *informed design* process [8].

Key words: digiskills, digital geometry and graphics literacy, informed geometry and graphics education

MSC 2020: 00A05, 00A66, 51N05, 01A05, 97U99

Figure 1: Digital Graphics as a multi-layered information process (image by author)

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Benefits and Downsides of e-assessment in Mathematics in Higher Education

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Assessment is one of the central components of a curriculum and guides students' learning. E-assessment involves the use of ICT to present and deliver assessment tasks, receive and record students' answers, as well as to provide feedback to students. E-assessment has been used considerably more in many higher education institutions since the onset of the COVID-19 pandemic and therefore our aim is to explore the specificities of e-assessment in mathematics.

We will present the benefits and downsides of e-assessment in mathematics in higher education. Our study focused on students' perspectives on e-assessment supported by a Learning Management System (LMS) in mathematics courses. It incorporated 631 students' responses to questionnaires conducted three times and it lasted for one academic year. We used descriptive statistics for close-ended questions and qualitative analysis for open-ended questions. The students' perspectives on e-assessment were generally positive. The students appreciated the student-centred approach based on sound pedagogical alignment of all teaching and learning elements with assessment. However, they found it important that technology should not distract them during assessment tasks. Finally, they did not find that there was more cheating and plagiarism in e-assessment in comparison to on-campus face-to-face assessments.

Key words: e-assessment, student perspective, cheating

MSC 2010: 97D60, 97B40, 97C70, 97–06

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Janos Bolyai's Angle Trisection Revisited

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J. Bolyai proposed an elegant recipe for the angle trisection via intersection of the arcs of the unit circle and of an equilateral hyperbola. It seems worthwhile to investigate the geometric background of this recipe and use it as the basic idea for finding the $n-$th part of a given angle, even so there will be no practical application for it. In this paper we shall apply this idea for the trivial case $n = 4$ and for $n = 5$.

Block Allocation of a Sequential Resource and Related Topics

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An H-packing of G is a collection of vertex-disjoint subgraphs of G such that each component is isomorphic to H. An H-packing of G is maximal if it cannot be extended to a larger H-packing of G. We consider problem of random allocation of a sequential resource into blocks of m consecutive units and show how it can be successfully modeled in terms of maximal P_m-packings. We enumerate maximal P_m-packings of P_n of a given cardinality and determine the asymptotic behavior of the enumerating sequences. We also compute the expected size of m-packings and provide a lower bound on the efficiency of block-allocation.

Minimal Invariant and Minimal Totally Real Submanifolds in Sol_0^4

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The space Sol_0^4 is one of the 19 model spaces of 4-dimensional Thurston geometries which belongs to non-Kähler model spaces. It is known that the Hermitian structures of all non-Kähler model spaces are locally conformal Kähler.

We consider minimal invariant and minimal totally real submanifolds in the 4-dimensional homogeneous solvable Lie group Sol_0^4 equipped with standard globally conformal Kähler structure.

We prove that the only minimal invariant surfaces of Sol_0^4 are totally geodesic hyperbolic planes. This is quite different from a situation in Kähler model spaces where holomorphic curves are automatically minimal.

We show that minimal totally real surfaces in Sol_0^4 tangent or normal to the Lee vector field are cylindrical surfaces.

Key words: 4-dimensional geometry, minimal submanifold, solvable Lie group, LCK structure

MSC 2020: 53C15, 53C25, 53C30, 53C55

Translation Surfaces with Constant Curvatures in 3-dimensional Lorentz-Minkowski Space

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Translation surfaces are surfaces generated by two curves moving along each other. In 3-dimensional Lorentz-Minkowski space, which is the smooth manifold $\mathbb{R}^3$ with flat Lorentzian pseudometric, such surfaces can be classified with respect to the causal character of their generating curves (spacelike, timelike or null (lightlike)). In this talk, we analyse translation surfaces with at least one null generating curve, which we refer to as null-translation surfaces. By considering generatrices as graphs of two functions with respect to the axis coordinate, we determine all null-translation surfaces of constant mean curvature and show that the only null-translation surfaces of constant Gaussian curvature are cylindrical surfaces. We also present alternative approach in analysing null-translation surfaces via Frenet frame, respectively null frame, of generatrices of the surface, that provides the explicit parametrization of null-translation surfaces with non-vanishing constant mean curvature.

Key words: Null-translation surfaces, Minkowski space

MSC 2020: 53A10, 53B30

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Fitting a Triangle on Three Lines

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We present a solution to the following fitting problem: Let g_A, g_B, g_C be arbitrary lines in space and let $A_0B_0C_0$ be some triangle. Find poses ABC of $A_0B_0C_0$ with $A \in g_A$, $B \in g_B$ and $C \in g_C$. We also characterize the configurations of g_A, g_B, g_C and $A_0B_0C_0$ admitting a motion where A, B and C run on g_A, g_B and g_C, respectively.

Key words: triangle fitting; motions with straight line paths

MSC 2020: 5108, 14N10, 70B10

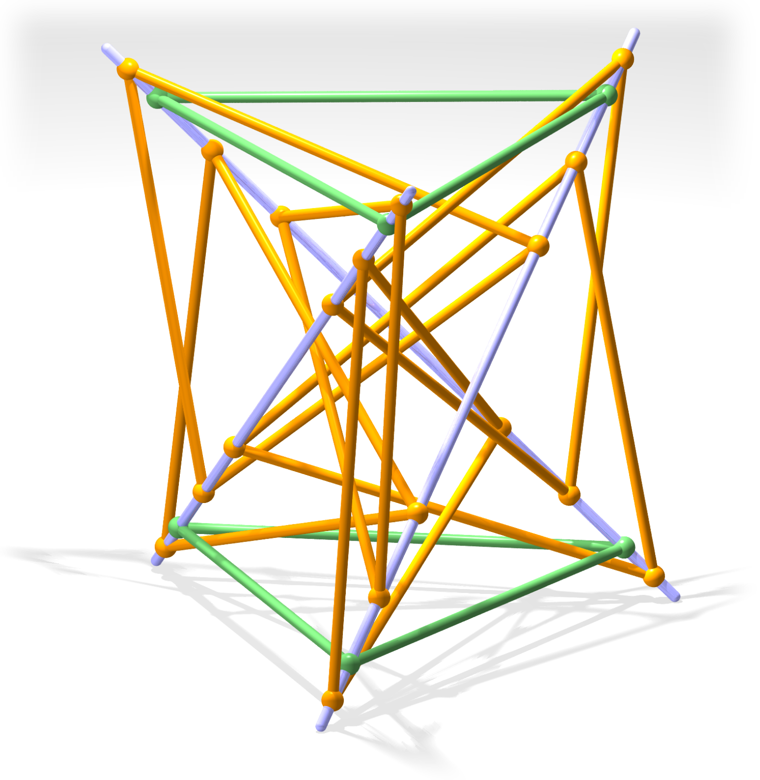

Figure 1: Fitting a triangle on 3 mutually skew straight lines. In this example the three given lines (blue) have rotation symmetry and the given triangle is equilateral. We obtain 8 solution triangles: 2 in planes normal to the rotation axis (green) and 6 other (orange).

Cross Science: Experience Geometry, Biology, and Physics. A Free Interactive Simulation Software

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Joint work with Franz Gruber

In the past 25 years, a geometric software package named "Open Geometry" was developed by several authors [4, 3]. This software was mainly used for internal purposes, i.e. illustrating books on geometry and mathematics. More than half a dozen such books with several thousand (!) illustrations were produced this way [1, 2, 5, 6].
However, the software additionally turned out to be a powerful tool for creating interactive, educational animations. Over the years, several people developed more and more such applications. Among them – in a prominent position – Franz Gruber, a former co-worker of the author, who unfortunately passed away recently.

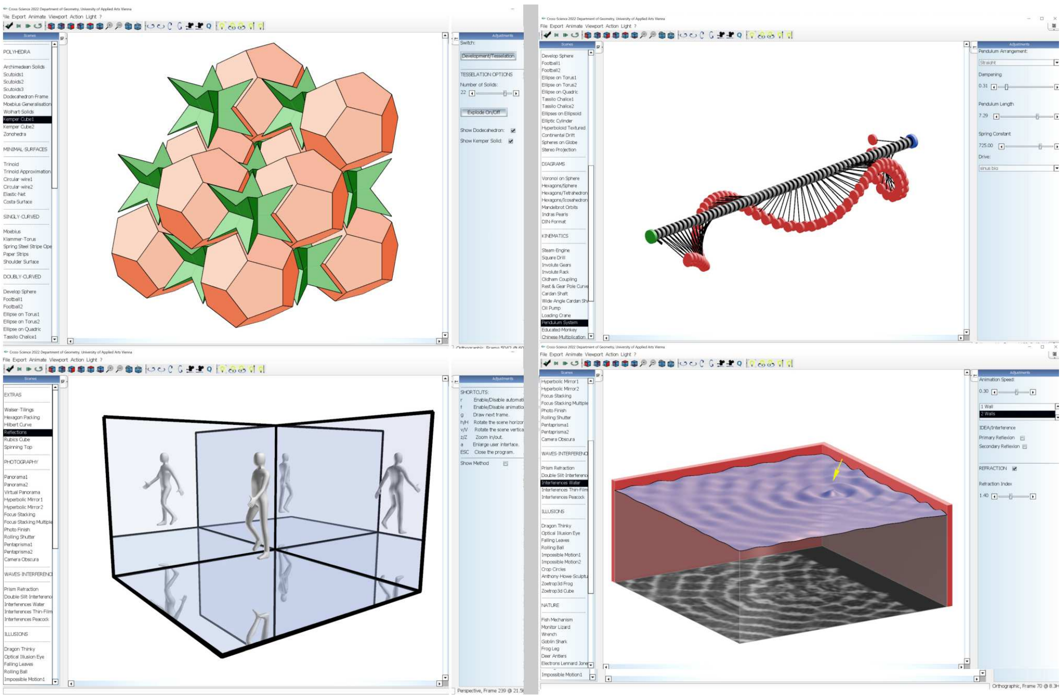

Figure 1: The surface of the "multiple executable" which – as an example – shows four different applications chosen by the user.

The software is now included in the publicly available pool for educational software. It contains more than one hundred interactive applications from the fields of geometry, biology and physics. Fig. 1 shows four examples of how this is done: One can select an application (left column). The corresponding executable is then launched (main screen) and can usually be manipulated via sliders or click buttons (right column).

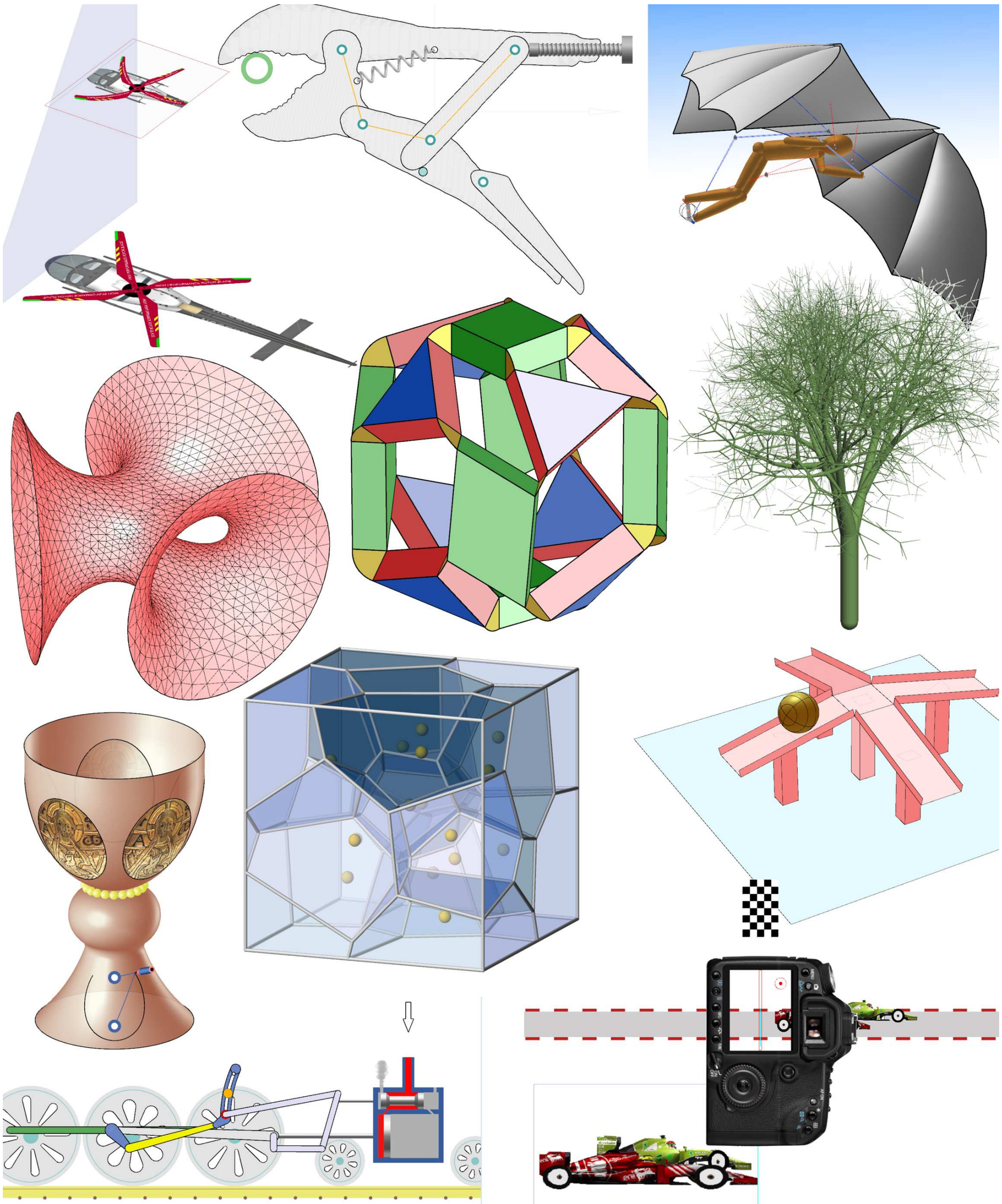

Figure 2: A collage of the output from 12 other applications shall illustrate the broad spectrum covered.

Of the many dozens of applications, the output of a dozen is compiled in a collage in Fig. 2. Since the theoretical background of many applications is quite sophisticated, a description of the algorithms is in progress and will be published by Springer Verlag in 2023.

Key words: educational software, interactive, animation, geometric illustration

MSC 2020: 97G99 , 97G80

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Mulitple Solutions of DirectKinematics of 3-RPR Parallel Manipulators

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A planar parallel 3-RPR parallel manipulator (Fig. 1) consists of three anchor points (A, B, C) in a base connected via three extensible legs (r_1, r_2, r_3) to a triangular platform (D, E, F). In the direct kinematics (DK) one has to compute the pose of the platform when the design of the manipulator (location of the base points and the shape of the moving platform) and the lengths of the legs are given. It is well known [1, 3, 4], that this task allows six solutions (Fig. 1 right, shows an example with six real solutions, i.e. geometrically, where the vertices of a given triangle are located on three given circles).

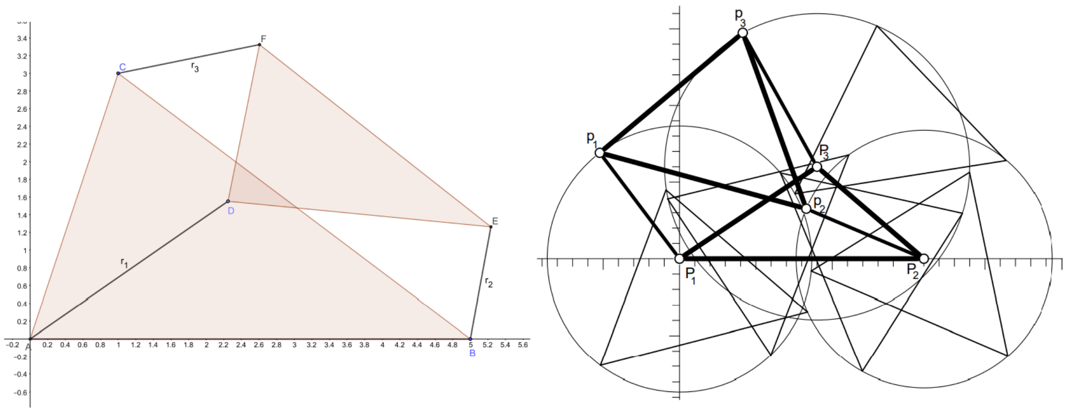

Figure 1: 3-RPR parallel manipulator   Example with six real solutions of the DK

When some of the solutions of the DK coincide the manipulator becomes singular (shaky) [5] or, from a different point of view, the framework of bars allows a flex ([6], where up to five coinciding solutions are treated). W. Jank, on the other hand, has shown that (symmetric) coupler curves can have curvature circles with six-fold tangency. Within this presentation we show how these two problems are related, generalize Jank's result to six-fold tangency of curvature circles for non symmetric coupler curves and give general conditions for the appearance of six coinciding solutions of the DK of planar parallel manipulators. Fig. 1 (left) shows an example with six coinciding solutions, Fig. 2 shows three fourbar linkages, related to this example, with a six-fold tangent curvature circle to their coupler curves.

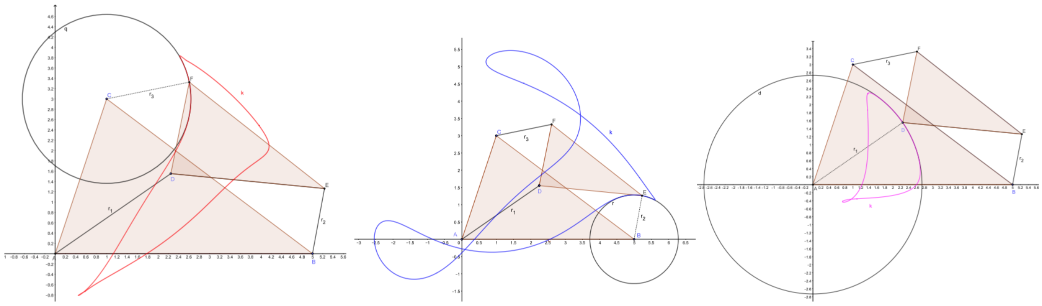

Figure 2: Three coupler curves, each with six-fold tangent curvature circle

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Loci of Centers in Pencil of Triangles in Isotropic plane

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We consider a triangle pencil in an isotropic plane consisting of those triangles that have two fixed vertices, while the third vertex is moving along a line. We study the curves of centroids, Gergonne points, symmedian points, Brocard points and Feuerbach points for such a pencil of triangles and prove the following statements:

Let the points A and B and non-isotropic line p be given. Let $\mathcal{T}$ be a pencil of triangles ABC such that C lies on p.

- The curve of centroids of all triangles $ABC \in \mathcal{T}$ is a line parallel to p.
- The curve of symmedian points of all triangles $ABC \in \mathcal{T}$ is an ellipse k_S.
- The curve of Gergonne points of all triangles $ABC \in \mathcal{T}$ is a rational curve of degree 3 passing through A and B.
- The curve of the Feuerbach points of all triangles $ABC \in \mathcal{T}$ is 2-circular rational curve of degree 3 having an ordinary double point at the absolute point.
- The curve of the first Brocard points (the second Brocard points) of all triangles $ABC \in \mathcal{T}$ is an entirely circular and rational curve of degree 4 with a cusp at A (B). It touches p at the point parallel to B (A), and the tangent line at B (A) is parallel to p.

In a special case when p is an isotropic line, the curves of centroids, symmedian centers, Gergonne points, Feuerbach points and Brocard points of all triangles ABC such that C lies on p are the isotropic lines.

Key words: isotropic plane, pencil of triangles, centroid, Gergonne point, symmedian point, Brocard points, Feuerbach point

MSC 2020: 51N25

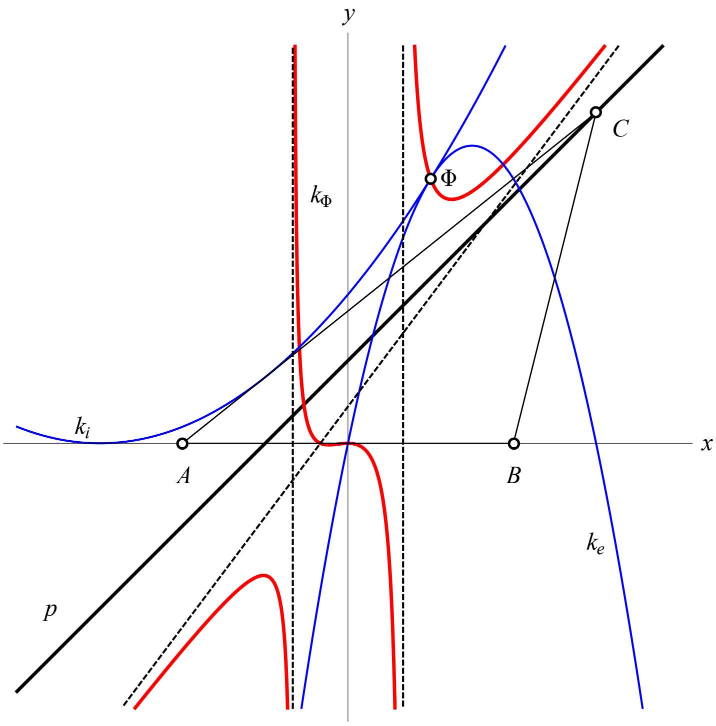

Figure 1: The locus k_Φ of Feuerbach points for the pencil of triangles with two fixed vertices A, B, and the third vertex C moving along the line p.

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Transcendental Reality

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Today we all know that there are more levels of infinity, the smallest, countable one and more of them beyond. But what we know today was discovered mere 200 years ago. Starting with G. Cantor and supported by one of the most prominent mathematicians D. Hilbert, who welcomed it and deeply involved it in his own accomplishments. First on his famous list of problems to be solved was *Cantor's Problem of the Cardinal Number of the Continuum.*

The notion of transcendental numbers and their applications both in analysis and in geometry is now well known, understood and commonly used.

In this talk I will present shortly the history of this subject and try to fit it into today's worldview. How does this deep, philosophical and I would dare to say religious concept of continuum reflect in our science, in our investigations or does it at all?

Key words: philosophy, geometry

MSC 2020: 51P99, 86A30

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On Properties of some Brocard Figures of a Triangle in the Isotropic Plane

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In this talk, we study some figures associated with a triangle in the isotropic plane. We present statements about Crelle-Brocard points, Brocard circle and the first Brocard triangle in the isotropic plane, and consider a number of significant properties of the introduced concepts. We explore relationships between the mentioned figures and some other objects related to the given triangle. We also consider relationships between the Euclidean and the isotropic case.

Key words: isotropic plane, Crelle-Brocard points, Brocard circle, first Brocard triangle

MSC 2020: 51N25

Inner Isoptics of a Parabola

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For a given curve and a given angle θ, the θ-isoptic curve [1] of a parabola is the geometric locus of points through which passes a pair of tangents to the parabola making an angle equal to θ. We explore the inner isoptics of parabolas: they are the envelopes of the lines joining the points of contact of the parabola with the tangents through points on a given isoptic. We show that the inner orthoptic (that is, the inner 90^{o}-isoptic) of a parabola is a degenerate point (the focus), but in other cases *the inner isoptic is an ellipse.*

Inner isoptics of ellipses have already been studied, at least partially, by the second author and Mozgawa in [2], and by Naiman, Skrzypiec and Mozgawa in [3]. In our contribution we study a simpler case, the case of a parabola. Our result connects parabolas, their "outer" and "inner" isoptics (hyperbolas and ellipses, respectively), thus we present a remarkable connection among different kinds of conics. We give an overview on how our GeoGebra experiments turned into conjectures, and a theorem, and how we found an analytical proof after implicitizing the problem via elimination and Gröbner bases.

Key words: conics, inner isoptics, implicit forms

MSC 2020: 53A04, 51M15

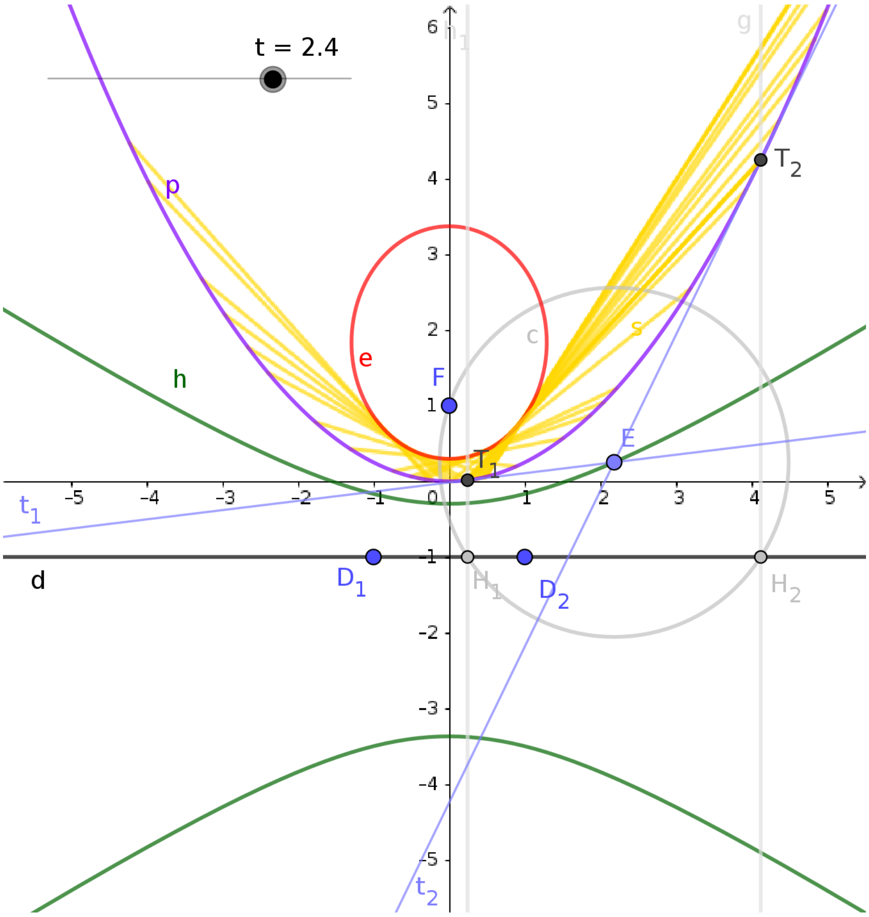

Figure 1: We consider the parabola p, defined by focus F and directrix $d = D_1D_2$, and its θ-isoptic where $t = \tan^2\theta$. Here the isoptic is the hyperbola h. An arbitrary point $E \in h$ and the tangents to p through E are labelled with t_1 and t_2. The tangent points are T_1 and T_2. Helper objects circle c and intersection points H_1 and H_2 allow us to analytically observe the situation. By denoting T_1T_2 by s, the envelope of s is the ellipse e.

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Remote Learning (at Descriptive geometry) and its Consequences

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In the life of every educator there are events that he does not plan and they mean a change in the way of thinking and teaching. How he will react to this kind of changes depends on several factors. Factors depend not only on the educator himself but also on the people he teaches, spatial capacities, technical capabilities and more.

The Covid pandemic over the past two years has certainly been a time that has brought about changes in the pedagogical field. In addition to the many problems that needed to be solved, it also brought certain innovations which makes sense to keep in the future. In this paper we want to present the transition to remote learning in the subject of Descriptive geometry at the Faculty of Architecture, University of Ljubljana.

Due to the good experience with the use of ICT, the transition to remote learning at the course was not difficult. At the Faculty of Architecture, we have a long tradition of evaluating spatial performance, therefore we wanted to check how distance learning affected the students. The results of the spatial ability test showed an increase of the level. The attendance of students on the lectures and exercises in that time was excellent.

But the return to the classic way of teaching has unfortunately proven to be more difficult for students. We have shown this by comparing colloquium results. There are probably several reasons for this situation, and it is probably not even possible to identify all of them.

The weak part of the remote learning is primarily sociological in nature. But it also has good qualities, such as optimization of student time for study. Remote learning also requires a different way of thinking and more self-discipline.

Key words: distance learning, Covid pandemic, descriptive geometry, consequences

MSC 2020: 97G80

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Realizations of Stasheff Politope by Means of Alternating Sign Matrices

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Joint work with Ivica Martinjak

A method of determinant evaluation known as the condensation has a natural interpretation in terms of the alternating sign matrices. An alternating sign matrix (ASM) is a square matrix of 0s, 1s, and -1s such that the sum of each row and each column is 1 and the nonzero entries in each row and each column alternate in sign. These matrices generalize permutation matrices. We study a family of ASMs with pattern avoidance and the property that the rightmost 1 in the row $i+1 \geq 2$ is to the right of the leftmost 1 in the row i, and further families with analogous properties. We will show symmetries among these families of matrices. Stasheff polytopes, also known as asociahedra, appeared in Jim Stasheff's work in the 1960s. A d-asociahedron is a convex polytope of dimension d whose i -dimensional sides are denoted by significant orders of $d-i$ pairs of parentheses between $d+2$ independent variables with appropriate incidence mapping. The vertices of the associahedron also correspond to triangulations of a regular polygon. We establish a one to one correspondence between the family of ASM of order $d+2$ and vertices of d-asociahedron.

Key words: alternating sign matrices, Stasheff polytope

MSC 2020: 52-xx

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Spatial Intelligence and Plane Sections

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Spatial visualization skills or spatial intelligence are essential to success in many disciplines. For engineering students, they might be decisive for their future career. Researchers recognized the importance of spatial ability and studies have been performed in the field of mathematics education, chemistry, physics education and psychology. Many studies have shown that there are correlations between various measures of spatial skills and performance in particular Science, Technology, Engineering and Mathematics.

Low spatial abilities can lead students to drop out of their university studies. The preknowledge and motivation differs a lot among the students. One of the main problems of the traditional teaching is the fact that these problems cannot be easily managed.

This report investigated the spatial visualization skills of first-year engineering students by using plane sections. The aim is to evaluate test results of engineering students, with special emphasis on Mathematics success and gender differences.

Key words: descriptive geometry, engineering, higher education, intelligence

MSC 2020: 51N05, 97M50, 97B40, 97C40

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Exeter Transformation

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Joint work with Peter Csiba

The Exeter point is one of the well-known triangle centers, number 22 among the centers listed in the online *Encyclopedia of Triangle Centers – ETC* [2]. The Exeter point of a given triangle ABC is defined from the centroid of the triangle by a drawing process [2, 3, 4]. Define A' to be the point (other than the triangle vertex A), where the triangle median through A meets the circumcircle of ABC, and define B' and C' similarly. The Exeter point is the perspector of the circum-medial triangle $A'B'C'$, and the tangential triangle $A_tB_tC_t$ (see Figure 1).

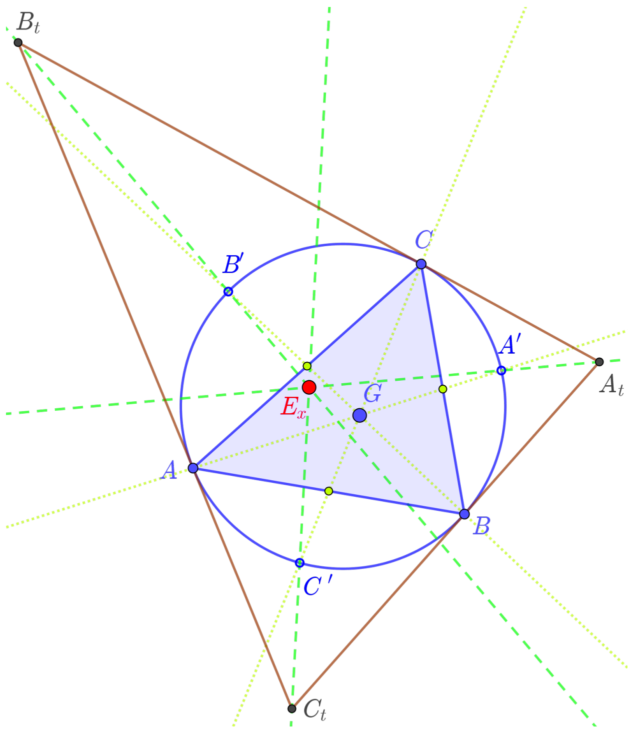

Figure 1: The Exeter point E_x

In Figure 1 the centroid of the triangle ABC is denoted by G and the Exeter point by E_x. The role of medians and their intersection (centroid) in the previous definition enables us to generalize: Let P is an arbitrary point in the plane ABC. Define A' to be the point (other than the triangle vertex A), where the line AP meets the circumcircle of ABC, and define B' and C' similarly. The triangle $A_tB_tC_t$ is the tangential triangle of ABC, so the lines A_tB_t, B_tC_t and C_tA_t are tangents of the circumcircle at points C, A and B, respectively. In this way from each point P we obtain its image P_e, the intersection of lines $A'A_t$, $B'B_t$ and $C'C_t$, and we call the process the *Exeter transformation* [1] of the plane with respect to the triangle ABC.

We show that a point P and its image $P_e = P_1$ (and its image $P_2, \ldots$) by Exeter transformation are on a well-defined conic $\mathcal{Q}(P)$ (see Figure 2), and we give some other properties of this conic and the Exeter transformation as well.

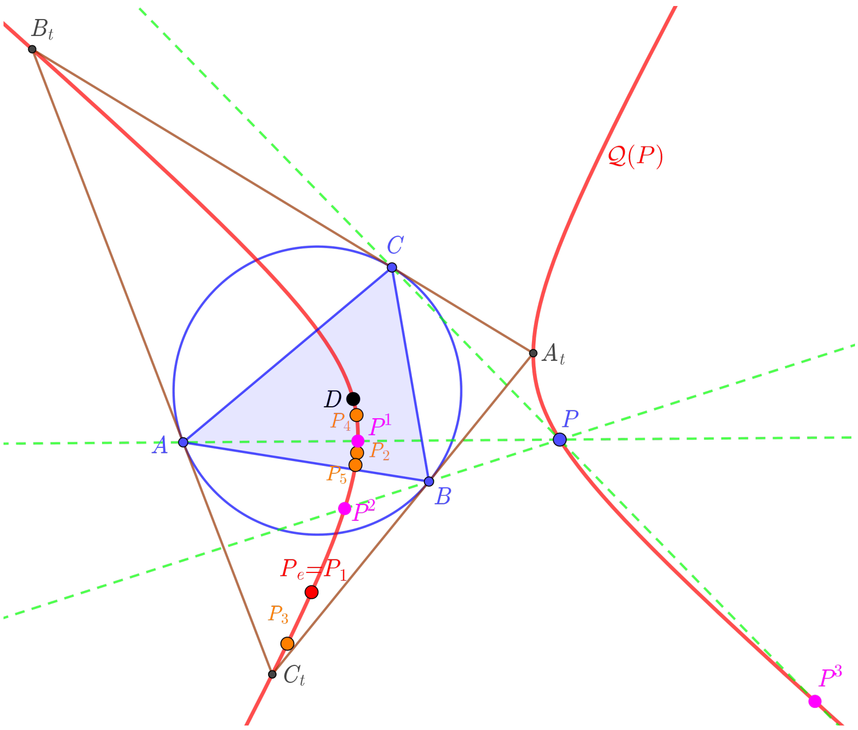

Figure 2: Points on the conic $\mathcal{Q}(P)$

Key words: Exeter transformation, Exeter point, barycentric coordinates

MSC 2020: 51A05, 51N15, 51N20, 97G40

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On Extrinsically Flat Surfaces That are Ruled and a Characterization of Riemannian Manifolds of Constant Curvature by Ruled Surfaces

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Special classes of surfaces play a fundamental role in geometry, where they help visualize key concepts and often allow us to test conjectures. Here, we are going to consider the class of *ruled surfaces*, i.e., surfaces foliated by a family of geodesics, and try to understand to what extent the behavior of this class determines the geometry of the ambient space. For a constant curvature, we first provide a model-independent proof for the known fact that extrinsically flat surfaces in space forms are ruled [1] (see, e.g., [2, 3, 4, 5] for model-dependent proofs). This allows us to identify the necessary and sufficient condition the curvature tensor of the 3d ambient manifold must satisfy for an extrinsically flat surface to be ruled [1]. Namely, we show that the curvature tensor satisfies $R_{trrn} = 0$ on the points of the ruled surface, where n refers to the direction normal to the surface, r refers to the direction of the rulings, and t refers to the remaining tangent direction. Using this characterization, we finally establish our main result [1]: if a 3d connected manifold has an extrinsically flat surface tangent to any 2d plane and if they are all ruled, then the manifold must have constant curvature. Finally, as an application, we prove that there must exist extrinsically flat surfaces in the Riemannian product of the hyperbolic plane, or sphere, with the reals and that they do not make a constant angle with the real direction [1].

Key words: ruled surface, space form, flat surface, extrinsically flat surface, constant angle surface.

MSC 2020: 53A05, 53A35, 53B20, 53C42.

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Poncelet Grids Associated With a Projective Billiard

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A projective billiard is a polygon with a circumconic and an inconic. Similar to the classical billiards in conics, the intersection points between the extended sides of a projective billiard are located on a family of conics which form the associated Poncelet grid. We extend the projective billiard by the inner and outer billiard and disclose various relations between the associated grids and their conics. Moreover, these Poncelet grids go along with remarkable configurations of lines.

Key words: ellipse, billiard, caustic, Poncelet grid, billiard motion

MSC 2020: 51N20, 52C30

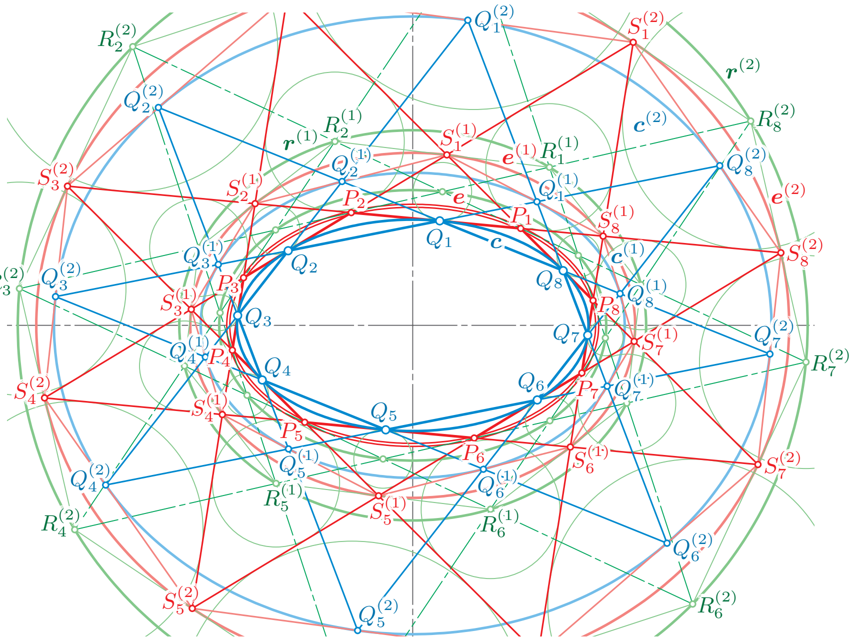

Figure 1: Periodic billiard $P_1P_2 \dots P_8$ (red) with the inner billiard $Q_1Q_2 \dots Q_8$ (blue) and the outer billiard (green) along with the associated Poncelet grids

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3D Printed Models Manufacturing and the Application of Concrete Manipulatives into the Educational Process to Support Learning and Teaching Geometry

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This contribution deals with the 3D printing and its uses in learning and teaching geometry at high schools and colleges.

First, several approaches to the producing of 3D models including their design, modelling and manufacturing will be presented. In the phase of designing a 3D model has to be described as a manifold 3D model, i.e., simply said, the object with a geometry that can exist in the real world. The modelling phase is based on the constructive solid geometry, on differential geometry, or on 3D scanning of real objects.

Second, the examples of working with 3D printed manipulatives to support teaching and learning of various geometrical topics at different levels of schools will be demonstrated.

Key words: 3D printing, constructive solid geometry, differential geometry, 3D scanning, learning and teaching geometry, 3D printed manipulatives

MSC 2020: 53A05, 51N20, 97G40

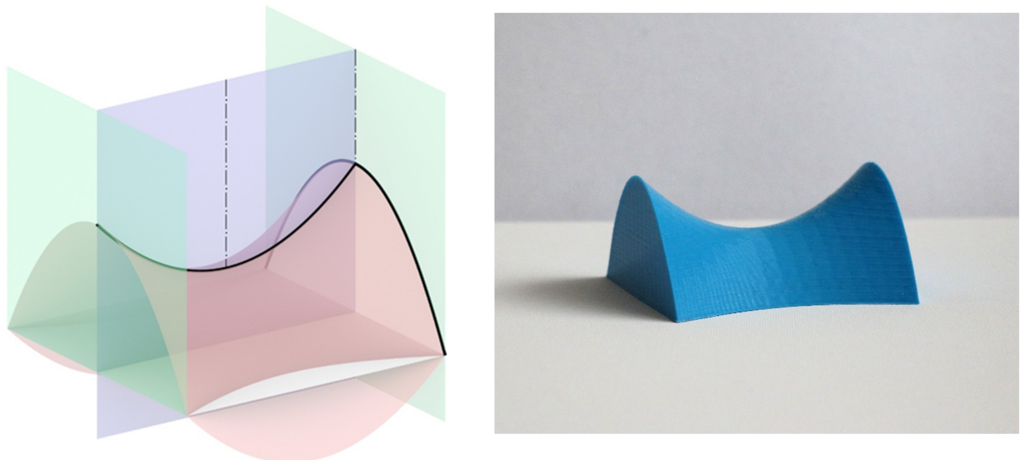

Figure 1: 3D computer model and 3D printed model of hyperbolic paraboloid

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Rational Curves with Pythagorean Hodograph

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We will discuss the relation between the polynomial and rational curves with pythagorean hodograph in $\mathbb{R}^2$ and $\mathbb{R}^3$ as well as the rational and polynomial pythagorean normal surfaces in $\mathbb{R}^3$.

The planar cases are considered rather for the sake of completeness and as a motivation. Indeed, the relation between the planar polynomial and rational PH curves was already fully analyzed in [2]. We will, however, compare these two families of curves using a different method based on solving a system of linear equations.

The situation is much more interesting in $\mathbb{R}^3$. Historically, the polynomial PH curves [3] are much better studied then the rational ones [5, 4]. On the other hand, the rational PN surfaces were fully described already in [9], but only examples of polynomial PN surfaces are available, see, e.g., [7, 1]

We propose a new method for studying these problems. It is based on determining the corresponding motion polynomial [8, 6]. While the primal (rotation) component of the motion polynomial is arbitrary, the dual (translation) part is determined be a linear system of equations. This system is analysed and possible denominators of the resulting PH/PN curves and surfaces are discussed. Polynomial objects in this approach appear as special cases of the rational ones. From a certain point of view, however, the polynomial objects appear to be the generic cases.

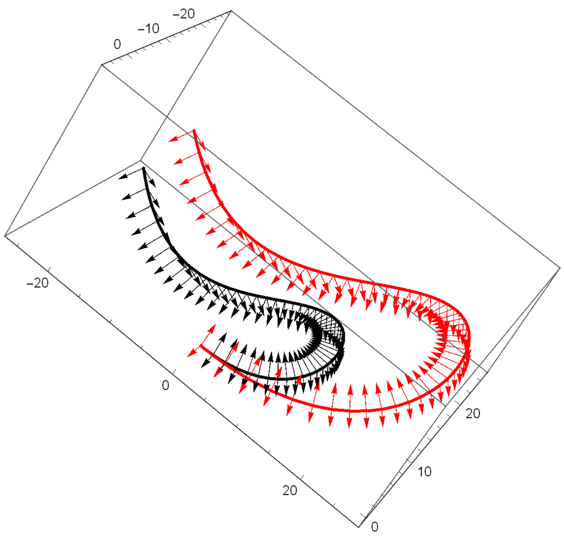

Figure 1.

Key words: curve, tangent indicatrix, residuum, interpolation, hodograph

MSC 2020: 53A04

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On the Separable Hadwiger Numbers of Platonic Solids

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Let K be a *d-dimensional convex body* ($d \geq 1$), that is, let K be a compact convex subset of $\mathbb{R}^d$ with nonempty interior. A *packing* is a collection of convex bodies in $\mathbb{R}^d$ whose interiors are pairwise disjoint. A packing is said to be *totally separable* if any two packing elements can be separated by an affine hyperplane of $\mathbb{R}^d$ disjoint from the interior of every packing element.

The *Hadwiger number* $H(K)$ of K is the maximum number of translates of K that all touch K and form a packing. $H(K)$ is also called the *translative kissing number of* K (see [1], [2]). The *separable Hadwiger number* $H_{sep}(K)$ of K is the maximum number of translates of K that all touch K and, together with K, form a totally separable packing (see [1]).

The Platonic solids are the five regular 3-dimensional polyhedra: the regular tetrahedron T_3, the regular octahedron O_3, the cube C_3, the regular dodecahedron D_3, and the regular icosahedron I_3.

On the Hadwiger numbers of Platonic solids, it is known that $H(T_3) = H(O_3) = 18$, $H(C_3) = 26$ (see [2]), $H(D_3) \geq 12$, and $H(I_3) \geq 12$. It is conjectured that $H(D_3) = H(I_3) = 12$. Based on the known Hadwiger numbers of T_3, O_3 and C_3, and on the corresponding extremal configurations of their translates, it is easy to see that $H_{sep}(T_3) = H_{sep}(O_3) = 18$, and $H_{sep}(C_3) = 26$.

We determine the separable Hadwiger number of the regular dodecahedron and the separable Hadwiger number of the regular icosahedron, proving $H_{sep}(D_3) = H_{sep}(I_3) = 8$.

Key words: totally separable packing, Hadwiger number, separable Hadwiger number

MSC 2020: 52C17, 05B40, 46B20

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Pselical Surfaces and Armiloid

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Pselical surfaces form a specific group of surfaces that can be regarded as special group of two-axial surfaces of revolution, namely surfaces of Euler type, see in [1], [2]. These surfaces can be generated by simultaneous revolution of a basic curve about two skew perpendicular axes in the space. Armiloid is one typical representative of this group of surfaces. Armiloid has been defined and presented by professor František Kadeřávek (1885 - 1961), an outstanding Czech geometer and mathematician, in his paper entitled *Generalisation of surfaces of revolution* that was printed in the scientific journal *Věstník Královské české společnosti nauk* [3].

Armiloid can be generated by a composite movement of a basic curve (a circle or an ellipse), which is composed from two systems of central collinear transformations determined by characteristics, while their axes are in two skew and perpendicular lines in the projective space. Furthermore, determining elements of the two collineations are in a special relation. Centres of one system of these central collineations are located always on the axes of the other system of central collineations involved, while characteristics of the two systems are inverse real numbers. Surfaces generated by such generating principle form the group of pselical surfaces.

An easy form of parametric equations of armiloid can be obtained by positioning the basic ellipse to the coordinate plane $v = xz$, its axes parallel to these coordinate axes with parameters $a, b, m \in \mathbb{R}$. Generating systems of central collineations are determined as follows:

$$\mathcal{K}^1(o^1 = z, S^1, h(v)), \quad \mathcal{K}^2(o^2 \parallel y, S^2, l(v)), \quad S^1 \in o^2, \quad S^2 \in o^1$$

where functions $h(v)$, $l(v)$ are C^1 continuous on unit interval $\langle 0,1\rangle \subset \mathbb{R}$ and define characteristics of the two systems of collineations. A system of homotheties is determined in the plane $\pi = xy$ with centres in the origin O and coefficients defined by C^1 continuous function $l(v)$ on interval $\langle 0,1\rangle \subset \mathbb{R}$.

Parametric equations of armiloid (pselical surface) on $\langle 0,1\rangle^2 \subset \mathbb{R}^2$ are

$$\begin{aligned} x(u,v) &= (m + a\ \cos 2\pi u)k(v)\ \cos 2\pi v \\ y(u,v) &= (m + a\ \cos 2\pi u)l(v)\ \sin 2\pi v \\ z(u,v) &= b\ h(v)\ sin 2\pi v \end{aligned}$$

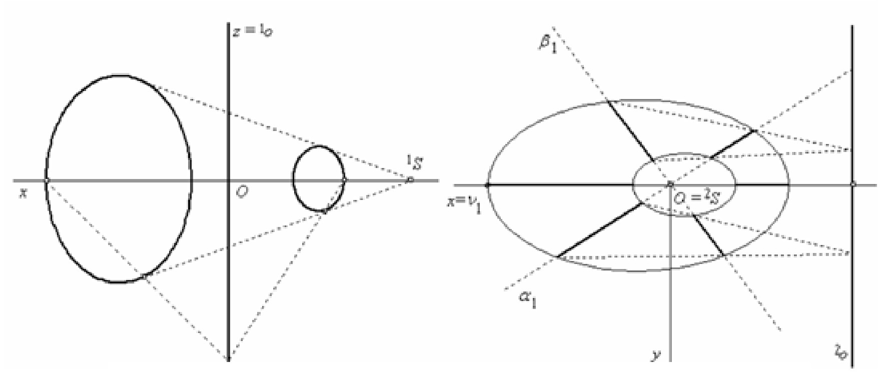

Figure 1: Generating central collineations in planes v and π and basic ellipse

Another related family of surfaces are spirical surfaces generated from the basic ellipse by family of affine tranformations, which can be regarded as generalised tori. Generating system of affinities are determined as follows: $\mathcal{A}^1(o^1 = z, s^1 \perp o^1, h(v))$, while system of homotheties is determined in the plane $\pi = xy$ with centres in the origin O and coefficients defined by function $l(v)$. Functions $h(v), l(v)$ are C^1 continuous on unit interval $\langle 0, 1\rangle \subset \mathbb{R}$.

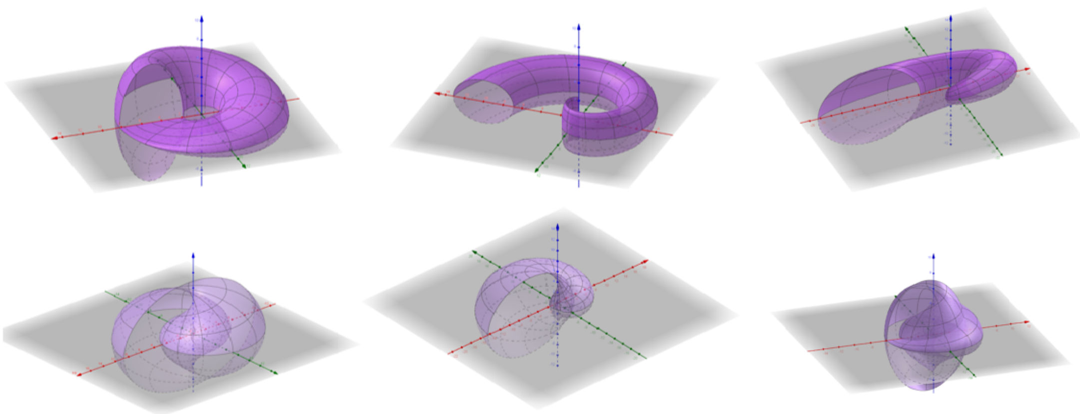

Figure 2: Basic forms of armiloid and pselical surfaces

Parametric equations of spirical surfaces on $\langle 0, 1\rangle^2 \subset \mathbb{R}^2$ are

$$\begin{aligned} x(u,v) &= (m + a\ k(v)\cos 2\pi u)\cos 2\pi v \\ y(u,v) &= (m + a\ l(v)\cos 2\pi u)\sin 2\pi v \\ z(u,v) &= b\ sin2\pi v \end{aligned}$$

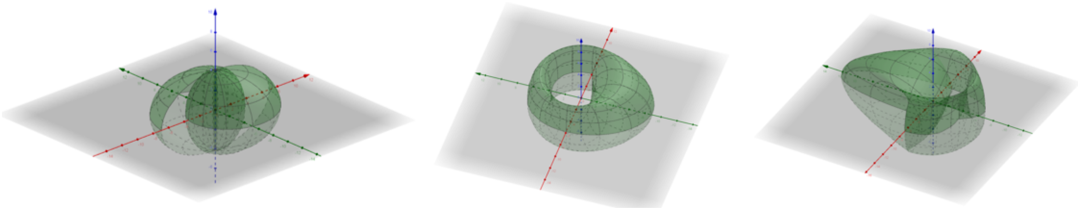

Figure 3: Various forms of spirical surfaces

Key words: modelling surfaces, generalised surfaces of revolution number

MSC 2020: 53A05, 53Z50, 65D17

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Games in Non-Euclidean Geometries

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When applied to video games, the term 'non-Euclidean' is not always used strictly in its mathematical sense. It is often employed to describe virtual worlds whose properties diverge from what we would expect of the world we live in.
There are, however, approaches, that successfully utilize properties of non-Euclidean geometry as a game mechanic both in two and in three dimensions. Additionally, the existing rendering pipeline of graphic cards allows for simple visualization of these geometries in 3D with only slight modifications making even virtual reality exploration of these geometries possible.
To highlight the differences of these geometries, in this work we want to examine how well classic games would translate from Euclidean to non-Euclidean geometries. Properties of the game play and rules are first examined in Euclidean space for games such as football or billiards. We then describe how these properties change and what rules would need to be adapted when entering spherical or hyperbolic geometry. Furthermore, we investigate challenges in their visualization and compare our findings to the existing approaches.

Key words: non-Euclidean geometry, video games, computer graphics

MSC 2020: 51M10, 5108, 00A66, 51-04

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Paul Miller Duality Arising in a Symmetrical View of Pappus' Theorem

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We want to explore some curious phenomena that appear when we view the classical Pappus' theorem more symmetrically. We start with three (red) points A_1, A_2 and A_3 on one (blue) line, and another three (red) points B_1, B_2 and B_3 on another (blue) line, and look at all six possible (green) Pappus' lines that can occur by taking the A and B points in all possible orders. It turns out that these six lines pass three at a time through two yellow points, as in the Diagram.

This configuration was studied in my online YouTube lectures on the Hexagrammum Mysticum at the channel Wild Egg Maths, and one of my viewers Paul Miller made the important observation that there is then a symmetry between the original two blue lines and the two yellow points, which in turn yields, remarkably, three new (brown) points on each of the original blue lines.

In this talk we will give an overview of some of the interesting formulas and cross ratio relations arising from this situation.

Key words: Pappus theorem, projective geometry

MSC 2020: 51N15, 51E15

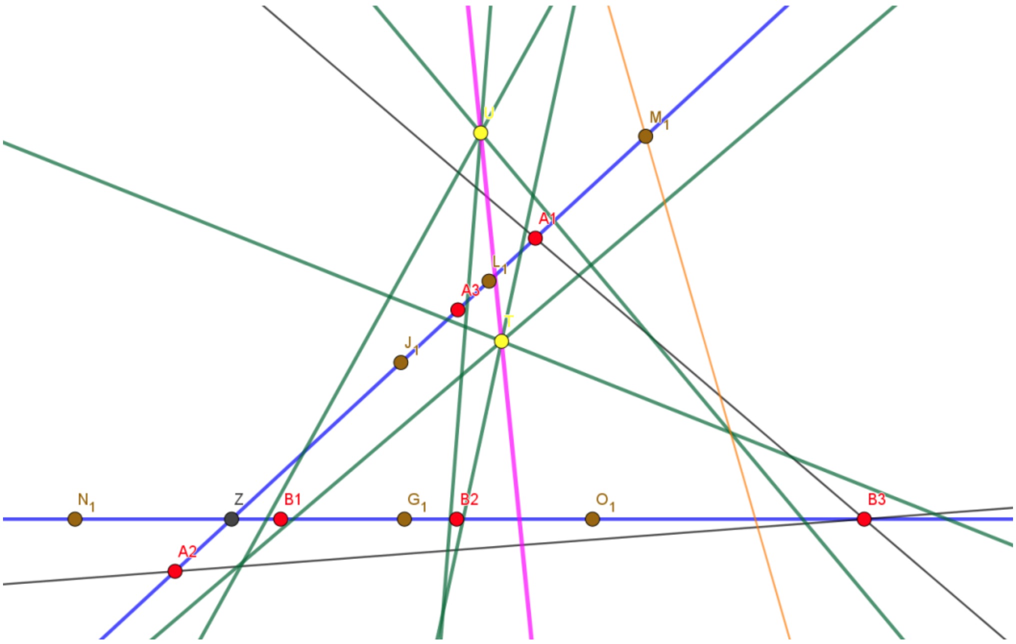

Figure 1.

Experimental Geometry: It Moves ... or Does It?

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Sometimes geometry and mathematics hinder each other because of their exactness and their rigid structure. Facts that are not exact are wrong and do not work. Because this approach works for many tasks, but for many it does not work, one has considered to build in a geometric or mathematical fuzziness. Here, geometric "configurations" that work only in a certain ε-environment are accepted as more or less correct.

Through the experimental architecture in our courses and research, we repeatedly encounter geometric challenges and issues that can sometimes be addressed more or less accurately through basic geometric knowledge. Sometimes, however, problems arise that go far beyond this. Since basic geometric knowledge is unfortunately less and less common in our clientele, often even simple questions are more than challenging. If these then become additionally complex and do not have an exact geometric solution, it becomes difficult and challenging for all involved, but of course also exciting.

In our presentation, we will talk about our latest geometric experiments and discuss our approaches to solving them. These include, the three-dimensional "moving" shapes derived from plane auxetics (Fig. 1 left) and double curved shells generated from curved tubes (Fig 1 right).

Key words: auxetics, movable systems, architectural geometry

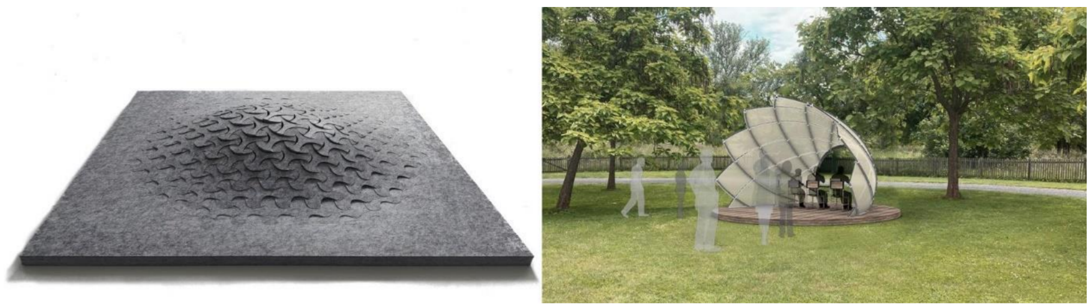

Figure 1: Auxetics and double curved shell

Rigid Body Displacement via Half-Turns

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Remarkably, any finite displacement of a rigid body can be achieved by rotation of 180° about each of two given axes.

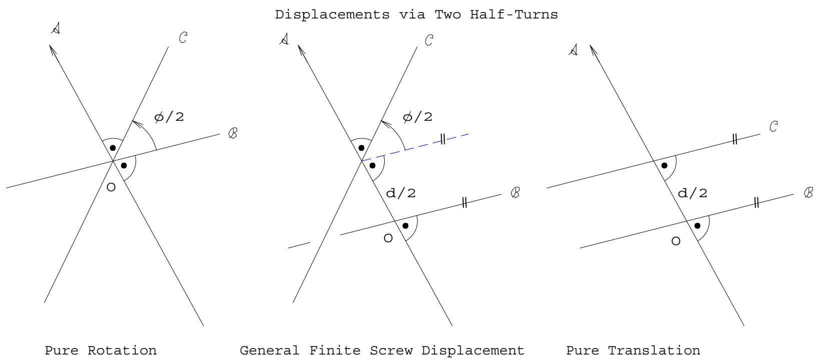

Figure 3: Displacement via Two Half-Turns about PB and PC

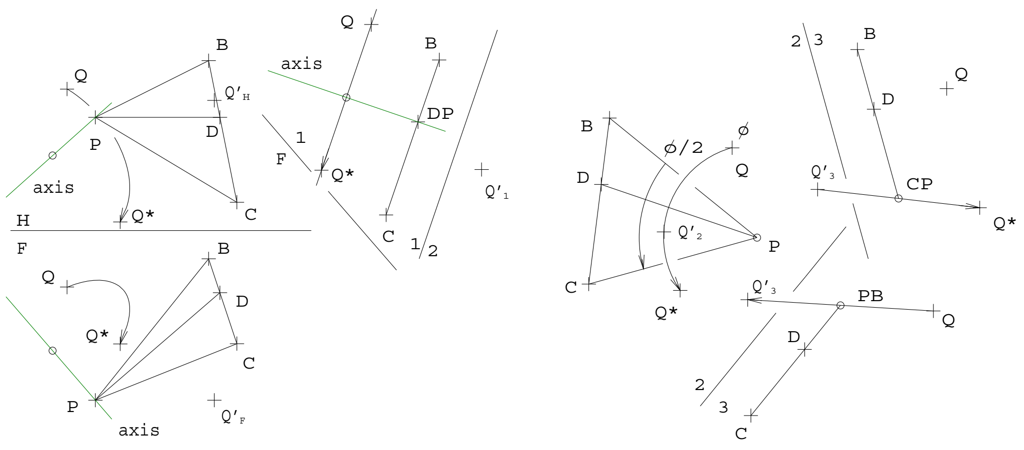

Figure 4: Rotation via Two Half-Turns about PB and PC

In Fig. 3 $\mathcal{A}$ is the screw axis, $\mathcal{B}$ is the first half-turn axis and $\mathcal{C}$ is the second. In pure rotation $R = P$. In pure translation absolute points on lines $\mathcal{B}$ and $\mathcal{C}$ are identical. In general displacement P, R, taken on $\mathcal{A}$, and absolute points on $\mathcal{B}, \mathcal{C}$, of lines on segments PB and RC, are distinct. Descriptive geometry in Figs. 4, 5 and 6 illustrate the effect by moving a point Q to $Q*$ via Q' after the first half-turn. Blaschke [1] introduced rotation via two half-turns (*Umwendungen*) while Jeger [2] showed that skew turn axes produce a finite screw displacement. His article was hard to find because he calls it "construction of spatial (line) congruence groups by reflection"; rather cryptic don't you think?

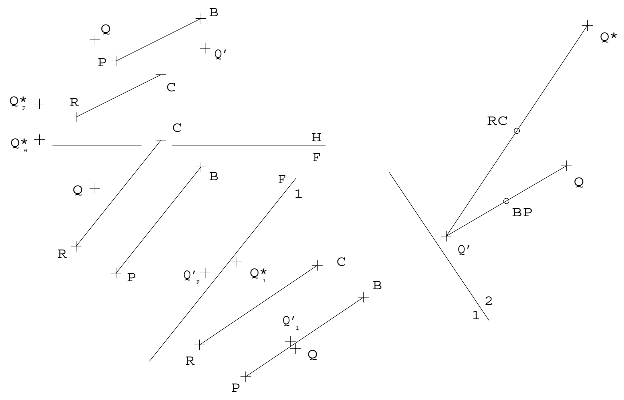

Figure 5: Translation via Two Half-Turns about PB and RC

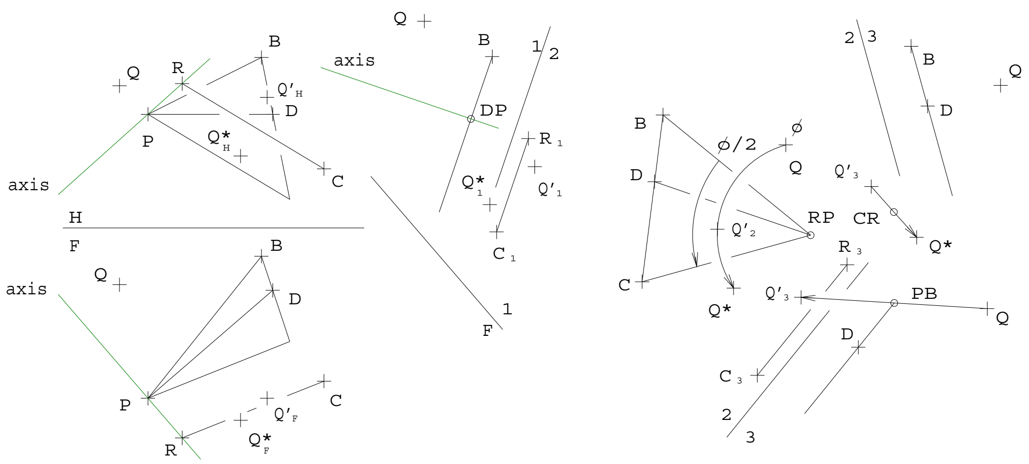

Figure 6: Screw Displacement via Two Half-Turns about PB and RC

Key words: rigid body, displacement, half-turn, rotation, translation, screw

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A Model of a Student Perspective on e-assessment in Mathematics in Higher Education

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Technology can support teaching and learning processes in many ways. In particular, e-assessment can provide a meaningful upgrade to standard approaches.

We will present a model of a student perspective on e-assessment in mathematics in higher education. The focus of our presentation will be on e-assessment supported by a Learning Management System (LMS). Within our mathematics courses paper-based assessments were replaced with computer-based ones. E-assessment questions were individualised, taken from a systematised online question bank, in which questions are organized according to a taxonomy of learning outcomes. E-assessments were conducted in a controlled online environment. We collected 631 students' responses to questionnaires on three occasions, in two different mathematics courses during one academic year. The research included two exploratory factor analyses and a confirmatory factor analysis. It yielded a model of a student perspective on e-assessment, consisting of four factors: Transparency and fairness of assessment, Formative and summative assessment and feedback, Meaningful use of technology in assessment and The difficulty of learning outcomes.

Key words: e-assessment, framework, student perspective

MSC 2010: 97D60, 97B40, 97C70, 97–06

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Posters

Implementation of Basic Geometric Transformations in the Construction Process

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In this overview, we show how perspective collineation (in a narrow sense) and its special affinity case are included in teaching geometry at undergraduate university studies of architecture. Presented constructions made in various contexts of descriptive geometry for future architects hide unavoidable steps as applications of these geometric transformations. We reveal them within a selection of examples.

Key words: perspective collineation, descriptive geometry, geometry education

MSC 2020: 97G80

The Benefits of Educative Curriculum Materials for Discovery-Based Learning in Primary Mathematics Education

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In active engagement with mathematical content and problems, learners acquire knowledge and skills in primary mathematics education. Their understanding is built up through their own activities and independent exploration of contexts. Such active-constructive learning goes back to Bruner's (1961) theory of discovery learning.

In order to implement this teaching concept, it requires a balance of independent and informative learning. Our research group, composed of mathematics professors and teachers and specialists in didactics and media design creates Educative Curricular Materials that can be used within the framework of discovery learning. The development already takes into account the new competency-based curriculum for Mathematics Education from the Austrian Federal Ministry of Education, Science and Research (BMBWF), starting in the school year 2023/24.

Educative Curricular Materials are teaching materials that have a dual function: They offer methodical-didactical structured activities for the learners and they promote the professional development of the teachers. The materials produced expand the teachers' subject knowledge and didactic repertoire.

In addition to the methodical-didactical correct structure and the integrated subject knowledge, further focal points in our project are the aspect of language awareness, the approach of Content Language Integrated Learning (CLIL-English) and the focus on material actions.

The next step of our research project is to test the materials in primary schools and evaluate them.

Key words: mathematics education, educative curricular materials, didactics of mathematics, discovery learning

MSC 2020: 97Dxx

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On Complete Quadrilateral in Rectangular Coordinates

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Joint work with Vladimir Volenec

A complete quadrilateral in the Euclidean plane is studied. The geometry of such quadrilateral is almost as rich as the geometry of triangle, so there are lot of associated points, lines and conics. Hereby, the study was performed in the rectangular coordinates, symmetricaly on all four sides of the quadrilateral with four parameters a, b, c, d. In this part we will study the properties of some points, lines and circles associated to the quadrilateral. All these properties are well known, but here they are all proved by the same method. During this process, still some new results have appeared. For example:
Parabolas circumscribed to trilaterals formed by three sides of the quadrilateral with axes parallel to the axis of parabola $\mathcal{P}$ inscribed to that quadrilateral arises from each other by using translations. They have parameter equal to the quarter of the parameter of $\mathcal{P}$.

Key words: Euclidean plane, complete quadrilateral, parabola

MSC 2020: 51N20

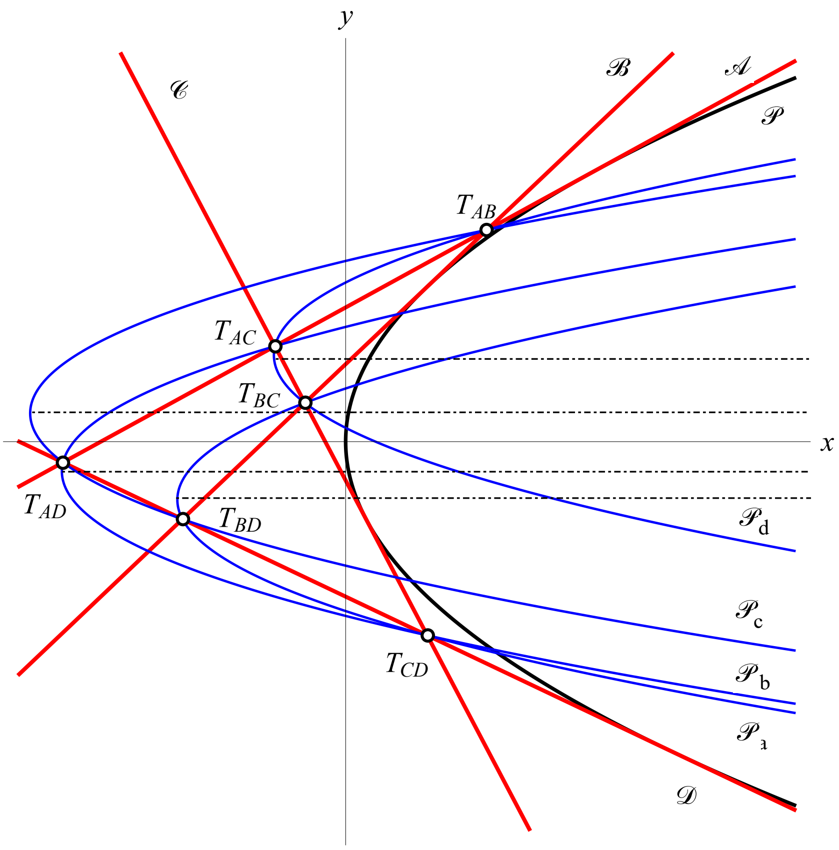

Figure 1: Parabolas circumscribed to trilaterals

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On a Coloring Problem in the Plane

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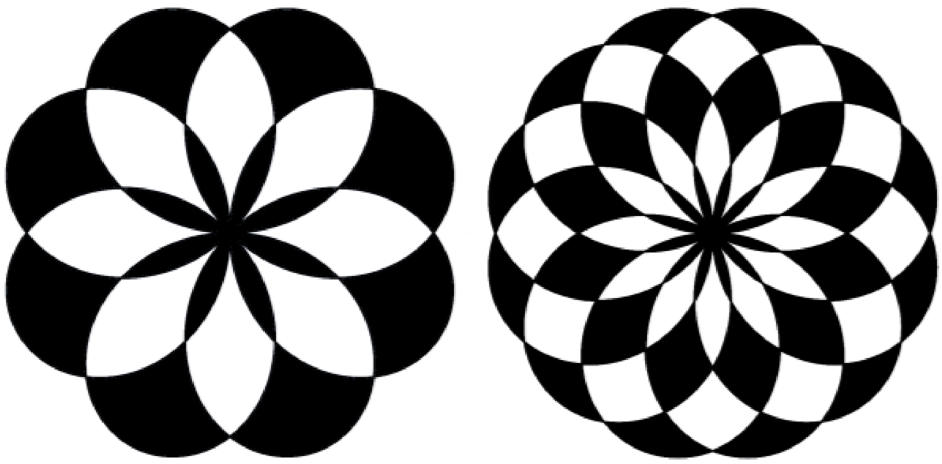
Joint work with Tomislav Došlić

Gregory Dresden of Washington & Lee University has mentored numerous undergraduate research projects [1]. In March 2022 he posted on his website the following problem:

Shown below (from left to right) are graphs of $r = \sin(4\theta/3)$ and $r = \sin(6\theta/5)$, where every other adjacent region (starting from the outside) is shaded black. Find the total shaded area for any such graph $r = \sin \frac{k+1}{k}\theta$, where $k > 0$ is an odd integer and θ ranges from 0 to $2k\pi$.



This also appeared in print as problem 1221 in the March, 2022 issue of the College Math Journal [2].

In this contribution we present an elementary solution to the problem and explore several possible generalizations.

Key words: rhodonea curves, rose curves, polar coordinates, coloring

MSC 2020: 51M25, 51N20

References

[1] `https://dresden.academic.wlu.edu/studentresearch/`

[2] Dresden, G., Problem 1221, *College Math. J.* **53** (2022), 152.

Generative Design of Surfaces

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Generative design is a process where the designer creates an algorithm that, with respect to the initial parameters, describes a certain model. Depending on curvature and smoothness, surfaces can be created in a variety of ways. We will describe some of the designs and algorithms. Algorithms will be created in *Grasshopper*, while the pictures of the models will be generated using *Rhinoceros*.

Key words: parametric modeling, applied geometry

MSC 2020: 65D17, 53A05, 14J29, 68U99

Helicoid and Twisted Towers

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Helicoid or a helical surface is a surface formed by simultaneous translation and rotation (screw motion) of a line around a fixed axis. This motion provides a circular helicoid which is also a ruled surface and a minimal surface. If we replace the line in the screw motion by any other curve we get a generalized helicoid. In civil engineering or architecture the screw motion mostly can be seen in staircases, but there are more and more buildings resembling a design that corresponds to the screw motion, e.g. “twisted towers”. We will represent some student's work from the elective course “Geometry in Civil Engineering” at the Faculty of Civil Engineering, University of Zagreb on this subject. The student models were made by the use of *Grasshopper* and cardboard models were fabricated by laser cutter.

Key words: helix, helicoid, parametric modeling, applied geometry

MSC 2020: 65D17, 53A05, 14J29, 68U99

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