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ABSTRACTS

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Plenary lectures

When Image sets Reality
Perspectival Alchemy in Velázquez's *Las Meninas*

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There are images in History of Art, Science, Technique, Humanities, which are milestones. *Las Menina*s is one of these. Several levels of reading and deepening have been proposed by historians and theoreticians, and as many interpretations have been made. Yet the very sophisticated construction of this masterpiece seems to escape any univocal hypothesis, making it as well astonishing as enigmatic.

Our interest in this extraordinary opera stems from the fact that it belongs to that special category of paintings whose meaning is inextricably based on and linked to their projective structure; therefore, aware of the wideness of the implications, we will mainly focus on its geometric and graphic feature.

Painted by Diego Rodríguez de Silva y Velázquez in 1656, this imposing painting (318 x 276 cm) shows an intriguing scene taking place in a lazy afternoon in the Galería de Mediodía of the Alcázar de Madrid. Strangely enough we basically see the painter in person at work beyond a large canvas, the Infanta Margarita Theresa of Spain, her Maids of Honour and other members of the Royal entourage, including a dwarf, a midget and a dog. Apart from the last two, all the characters are fixedly looking at us with curiosity, while we are looking into their space, that is, quite unusually, the studio of the painter. In this dialog of glances, looking more closely and attentively we surprisingly discover the image of the King Philip IV and Queen Mariana, reflected in a small mirror hanging in the middle of the rear wall, also they looking towards us. Above, two big mythological paintings appear, showing the hard punishments that await those who dare to challenge gods.

Any attempt to undertake a philological reconstruction of the geometric space of the scene has to deal with at least two issues. On the one hand the sophisticated perspective pattern of the painting, including the representation of mirrors and reflections effects, and the crucial fact that the large canvas appearing in the painting, slightly rotated and tilted, faces the artist, therefore its figurative content remains a secret. On the other hand, the fact that the real room represented in the painting was located in the south western wing of the castle, known as Cuarto del Príncipe Mejor, which burned during the big fire of Christmas Eve in the year 1734. On the bases of the available information, we aim to compare the presumable shape of the painted room as it results from the perspective reconstruction based on the depicted space, with the presumable shape of the real space hypothesized on the base of some historical plans, especially that of Juan Gómez de Mora (1626), and of some information about other important changes supervised by the same Velázquez afterwards. In this process, the reflection in the mirror will provide a valuable aid for the geometrical reconstruction, but at the same time it will set a limit to the investigation.

In conclusion we have no other chances but agree with Jonathan Brown's opinion: "A painting as rich in ambiguity as it is in subtlety, *Las Meninas* has long been recognized as a masterpiece of Western art, a pictorial tour de force rarely equalled and never surpassed. But when we attempt to explain its greatness, we soon realize how it seems to evade the grasp both of intuitive and rational understanding".

Nevertheless, the prospective meager expected results are largely offset by the wealth of experience resulting in the geometric study and graphic analysis of this painting, which is not only a masterpiece of genius, but also a superb educational example of scientific and artistic dedication as well as of professional ennobling.

Key words: Diego Velázquez, Alcázar de Madrid, perspective, geometry and graphics, photogrammetry, projective geometry, descriptive geometry, optics, catoptrics

MSC 2010: 00A05, 01A05, 51N05, 97U99

Figure 1: *Las Meninas* (underneath) and detail of the *Galería de Mediodía* (overlying). Diagram by the author.

Acknowledgments: I would like to express my thanks to Professor Maria Grazia Sandri for helping me with the historical sources and Professor Dario Angelo Maria Coronelli for reviewing the English text.

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"Math-Letters" from a Geometrical Point of View

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The Austrian Mathematical Society (ÖMG) publishes monthly the so-called "Math-Letters" (`http://www.oemg.ac.at/Mathe-Brief/`). They are mainly addressed to math teachers. The topics are, e.g., proposals for pre-scientific papers of pupils, development and history of different mathematical fields, biographies of mathematicians, etc.
So far, the author has already contributed six such letters. Being a geometer by education and heart, such letters have of course a specific "geometrical touch".

The topics of the six letters are:

1) Volume and surface of a sphere, and barycenters of solids, following the ideas of Archimedes (Figure 1).
2) Application-oriented examples of vector calculus (Figure 2).
3) Magic of reflections (Figure 3).
4) Proportions – a tool for the understanding of many mathematical questions (Figure 4).
5) Some magic of numbers (Figure 5).
6) Space collineations in photography and stereoscopy (Figure 6).

The talk will be a plead for the use of figures and common sense when explaining allegedly "not-so-easy-to-explain" mathematical questions. To give an example: Multiple reflections can be quite tricky, and even the case of two parallel mirror planes is not trivial at all.

Key words: math education

MSC 2010: 97D99, 97U10

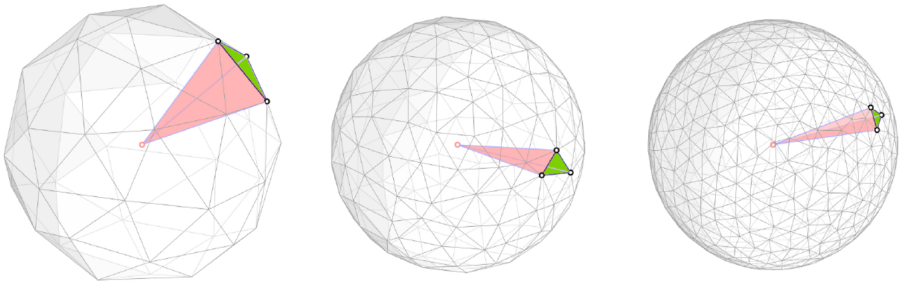


Figure 1: How to deduct the sphere's surface from its Volume

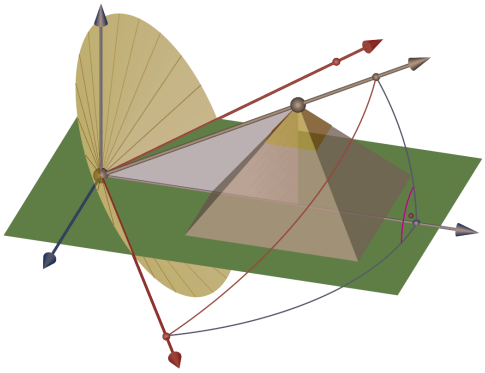


Figure 2: The calculation of sunrise and sunset (here in connection with the Great Pyramids) leads to the "spherical Pythagoras"

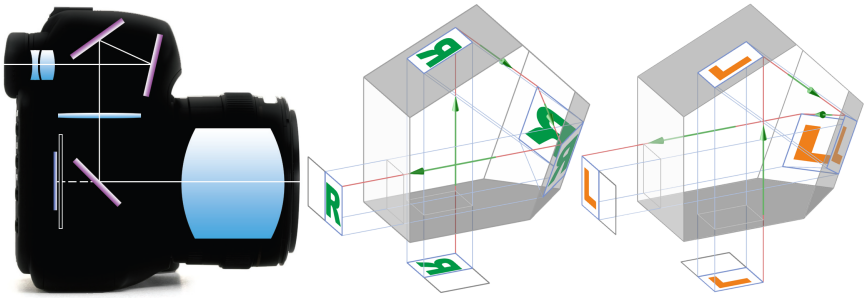


Figure 3: Multiple reflections in a roof pentaprism of a DSLR

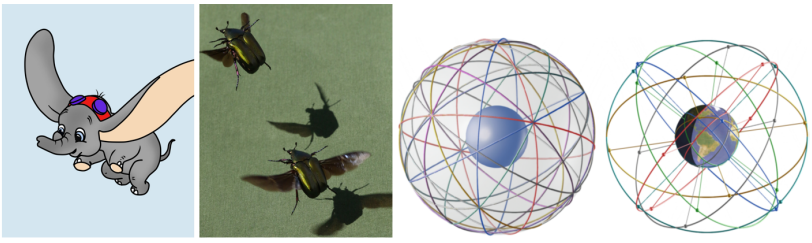
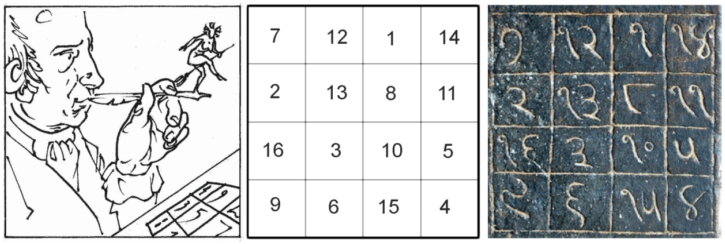


Figure 4: Left: Scaling in nature and its impact. Right: understanding Keplers third law.



7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Figure 5: Left: The Witches One-*Times*-One (J. W. v. Goethe) leads to the only magic square of order 3. Middle and Right: Famous magic squares of order 4

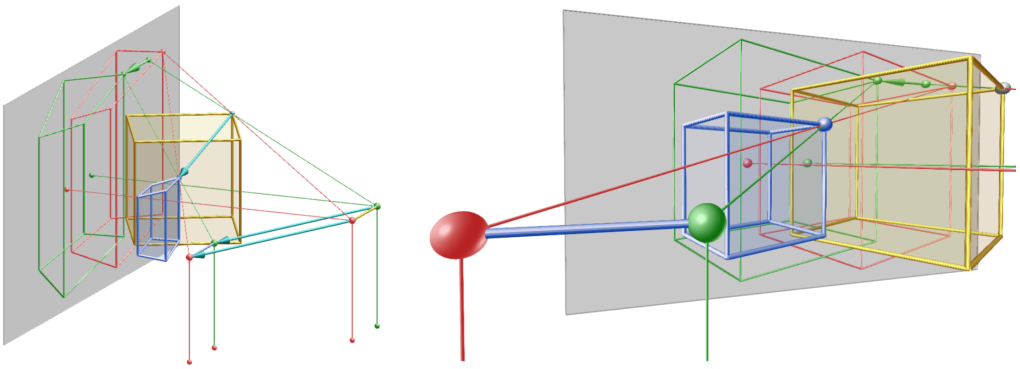


Figure 6: Stereoscopic viewing: The brain sees collinear objects

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Geometry Education and a Model of Geometry Competences

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I would like to present a project related to geometry education in primary schools. Children in primary school learn fundamental skills. These skills are the basis of their further education. I focus on a very particular part of their education: geometry. The value of geometry education seems to be significantly underestimated. One reason could be that since their first day on Earth all human beings share the perception to live in a three-dimensional space. It is such a basic experience that it can easily be overlooked what geometry instruction can do. In Austrian primary schools geometry education is included in mathematics education. It is common conviction that basic arithmetic, i.e., to be good at sums, is very important. This belief implicates that the primary curriculum, textbooks as well as primary teachers allocate a lot of room for calculating practice, and geometry classes are pushed on the margins merely considered hands-on activities. It is the aim of the project to illustrate the relevance of structured lessons on geometry from pre-primary to secondary school.

According to the primary curriculum some geometric knowledge is expected at the beginning of secondary school. A project named *Geometriekoffer*, literally translated *Geometry Case*, should ease the transition from primary school to secondary school. The project was started in 2009, initially by a small group of secondary school teachers and university teachers. As a first result, the group provided elementary school teachers with material and exercises for stimulating geometry instruction. Additionally, further education was offered to acquaint the primary school teachers with the material and also some geometric background. In 2012 we established a new subgroup which includes kindergarten teachers and teacher students. The material from the Geometry Case was tested in kindergarten groups. What the children loved most were the edge side riddle (a bunch of cuboids), and the geoboard (a square board with a total of 25 pegs). We developed exercises and tasks for geometric instructions in kindergarten (pre-primary education). Series of examples for the edge side riddle and the geoboard with increasing level of complexity were designed.

In order to systematically develop material for geometry instruction in primary school or kindergarten we need to know more about children's geometric thinking. At that point it was a natural step to integrate a theoretical approach to the topic. To analyze the children's geometric thinking we developed the *Model of Geometric Competences*. It can be visualized as a two-dimensional model (Figure 1). We call the first dimension *Spatial Thinking*, divided into three competences, listed in hierarchical order: Spatial Perception, Spatial Imagination, and Spatial Performance.

Model of Geometric Competences – Dimension Spatial Thinking:

- Spatial Perception means the orientation in the real world with all our senses, e.g. visual, auditory, and kinesthetic sense.
- Spatial Imagination means the capacity to remember what was seen, or to create ideas to something that has been heard.
- Spatial Performance means the ability to change the contents of spatial imagination.

However, the competence of spatial thinking cannot be monitored directly. Therefore, we constitute the second dimension, called *Expression of Spatial Thinking* as a means of demonstrating the basic competence of spatial thinking. Again, that second competence is divided into three segments.

Model of Geometric Competences – Dimension Expression of Spatial Thinking (short: *Spatial Expression*):

- Acting in 3D-Space (e.g. placing objects in the three-dimensional space).
- Acting in a 2D-Plane (e.g. drawing or sketching objects in a two-dimensional plane).
- Communicating (e.g. understanding and talking about geometric objects using an appropriate active and passive geometric vocabulary).

The third dimension in Figure 1 signifies that the model is open for further development. The model of geometric thinking can be used as a tool to analyze a task or an exercise. On the other hand, the model delivers information on knowledge and skills of children, and on the children's level of progression. Small activities where the teacher checks for basic skills at primary school entry, and formative assessments where the teacher checks for understanding are applications of the model.

Another model describing the progression in learning geometry is the *van Hiele Model of the Development of Geometric Thought*. It describes five hierarchical levels of geometric understanding. The most basic level is visualization - students recognize geometric objects by appearance, the second level is analysis - students see geometric objects as a collection of geometric properties. These two levels are applicable to the description of geometric understanding in kindergarten and in primary school. Therefore, it makes sense to compare the van Hiele model and the model of geometric competences.

At level 5 (Rigor) of the van Hiele model - students understand logical aspects of deduction, such as defining and comparing axiomatic systems, e.g. Euclidean and non-Euclidean geometries. Therefore, the van Hiele model can be used to describe students' achievements in geometry on a higher level. Both models are used as tools for geometry didactics. Additional research work is necessary to further improve the model of geometry competences.

Until now, the working group has been ever-expanding. The outcome evolved from a thriving collaboration between theory and practice. It is meant to support geometry education from pre-primary education to secondary education and to facilitate the transition between the different levels.

Key words: geometry education, primary school, secondary school, transition, model of geometric competences, van Hiele model, special didactics

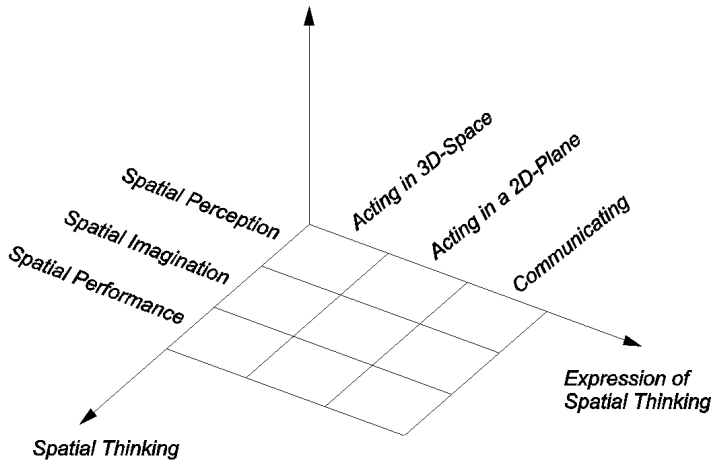

Figure 1.

Constructions on the Absolute Plane to the Memory of Julius (Gyula) Strommer 1920-1995

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Professor Julius (Gyula) Strommer was a charismatic figure of The International Society of Geometry and Graphics and also of our societies in smaller circle, surrounding countries of Hungary, from the beginning. On his anniversaries I would like to remind you on his main scientific activity in the field of Foundation of Geometry, or in particular, of geometric constructions on the absolute plane.

For this reason I remember you shortly on his life. He was born in Nagyenyed, Transylvania (in Romania nowadays). His talent led him to astronomy, to engineering and to geometry: on the tradition of the two Bolyai's. Especially he arrived to the absolute geometry, where the axiom of parallels of Euclid is not required. He reached a great reputation as a researcher, excellent teacher and for his human values. He also became the dean of the Faculty of Mechanical Engineering of the former Technical University of Budapest.

His main research topic will be summarized in this survey on the base of his academic doctor dissertation: To the theory of geometric constructions, independent of the axiom of parallels (in Hungarian, cited below). He starts with the Mohr-Mascheroni constructions exclusively by compasses. By the ideas of the "second Euclides Danicus", J. Hjelmslev, he extends these to the absolute plane, not using the projective embedding, in general, etc. His bravourous constructions are remarkable nowadays as well.

The author of this presentation, who was so lucky to be his scientific aspirant in the year 1974, could extend this projective machinery to the general circle geometry on the base of reflections in the sense of Friedrich Bachmann. My mentors Ferenc Kárteszi and Julius Strommer knew him personally.

Key words: absolute plane, geometric constructions

MSC 2010: 51B10, 51N15, 51F15

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On Symmetry Classification of Serbian Medieval Frescoes Ornaments

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Ornaments are the oldest and basic decorative elements in visual arts, closely connected to the concepts of repetition and symmetry. During the entire historical development of humanity, there existed unbreakable connections between geometry and art, where the visual presentation often served as the basis for geometrical consideration. Relatively independent development of geometry and painting resulted in formation of two different languages that use completely different terms for describing symmetrical forms.

The approach to the classification and analysis of ornaments based on symmetries was enriched by the contributions of different authors such as A. Müller, A. O. Shepard, N. V. Belov, D. Washburn, D. Crowe and B. Grünbaum. It offers possibilities for the more profound study of the complete historical development of ornamental art, regularities and laws on which constructions of ornaments are based, as well as an efficient method for the classification, comparative analysis and reconstruction of ornaments. The classification of ornaments according to their symmetries can help us to find answers to many questions: when and where a certain kind of symmetry appears in the ornamental art; which forms prevail; how to classify colored ornament; how, when, where, and why man created ornaments at all. This kind of classification can also be used as the indicator of connections between different cultures.

We will analyze ornaments from the exhibition "Memory Update - Ornaments of Serbian Medieval Frescoes", The Museum of Applied Art (Belgrade, November 6, 2013 - January 31,2014) based on the material presented in the book by art historian Zagorka Janc, "*Ornaments in the Serbian and Macedonian frescoes from the XII to the middle of the XV century*", Belgrade, 1961. The mathematical approach to the symmetry of ornaments is different from the art historical one, since the mathematicians try to show representative examples of particular symmetry groups. The ornaments from the Serbian medieval frescoes belong to the religious decorative art with the very restricted set of possible motifs, mostly related to a cross, and thus the basic motifs can fit in a very limited number of symmetry groups. The main criterion for the quality of such ornamental art could be the richness and variety of patterns obtained from a very small number of symmetry groups, proving the creativity of their authors - their ability to create variety with a very restricted number of initial symmetry groups.

Key words: ornaments, geometry, symmetry, symmetry group, classification

MSC 2010: 20C30, 14L35, 14L40, 51N30

Contributed talks

Architectural Shapes and Curvilinear Perspectives

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Since ancient times one of the basic issues has been to represent the living or lifeless objects of three-dimensional real world on a mostly planar surface, as e.g. wall of a cave, later on paper and nowadays on screen. From pictures made for sacral purposes became later the art; however, the construction of e.g. buildings required practically useful representation of objects. With the "common denominator" called geometry, art and engineering interrelated all the time, especially in architecture.

In this talk we try to find a new link between architectural shapes and geometrical principles. We analyze some ancient and new architectural elements from entasis and curved stylobate to modern buildings of G. Bunshaft and others, and try to understand them on the base of curvilinear perspective systems. We also refer to optical illusions that may influence the effect.

MSC 2010: 00A67, 91E30, 97G80, 97M80

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Pedal Curves and their Envelopes

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In the Euclidean plane with the pedal transformation the pedal curve of a given generating curve with respect to the pole can be obtained. The pedal curve c_e of a generating curve c_1 with respect to a pole P is the locus of the foot of the perpendicular lines from P to all tangent lines of the curve c_1, [4]. If the generating curve c_1 is a conic then its pedal curve c_e can be given as an envelope of circles, [2]. The pedal transformation can be extended in the quasi-hyperbolic plane where the metric is induced by the absolute figure $\mathcal{F}_{\mathbb{QH}} = \{F, f_1, f_2\}$. In the quasi-hyperbolic plane the pedal curve c_{qh} of a given generating curve c_2 with respect to the polar line p is the locus of the lines joining the points of the curve c_2 with its corresponding perpendicular points on the polar line p, [1]. In this presentation we will give the construction of the envelope of the pedal curve and study the pedal curve as an envelope of circles in the quasi-hyperbolic plane.

Key words: pedal transformation, pedal curve, envelope, envelope of circles, quasi-hyperbolic plane

MSC 2010: 51A05, 51M15, 51N25

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Geometry for Learning Analytics

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Learning analytics (LA) deals with the development of methods that harness educational data sets to support the learning process. It includes the measurement, collection, analysis and reporting of data about learners and their context. Learning analytics are used in order to interpret data about students' learning, assess their academic progress, predict future performance and personalize educational process.

Assessment of students is of fundamental importance. Assessment deeply influences learning, but at the same time assessment data are rarely considered as a part of available data utilized by learning analytics. One of the reasons for that is that available data is not granular enough [1]. Research on indicators and metrics that can be used in that context, especially on reliability and validity of assessment, peer-assessment and self-assessment, is (currently) very limited.

Inspiration for the development of new indicators and metrics can come from non-Euclidean geometries ([2], [4]) and also from multicriterial decision making [3].

Key words: non-Euclidean geometry, learning analytics, assessment, multicriterial decision making

MSC 2010: 97M70, 53A35

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Spherical Surfaces in Euclidean Space

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In the real projective space $P^3(\mathbb{R})$ the Euclidean metric defines the Euclidean space $\mathbb{E}^3$ with the absolute conic given by the equations:

$$x_0 = 0 \text{ and } A_2 = x_1^2 + x_2^2 + x_3^2 = 0.$$

While the circular curves in the Euclidean plane were investigated and obtained results were presented in many papers and books, the surfaces in $\mathbb{E}^3$ with the similar properties, and the order greater than 4, were in some way neglected. Motivated by this fact, we introduce so-called *spherical surfaces*. An n-order surface S_n of Euclidean space is called $p-$*spherical surface* if it contains the absolute conic as a p-fold curve. Every such surface is given by the following algebraic equation:

$$A_2^p\, g_{n-2p}(x_1, x_2, x_3) + \sum_{j=1}^{p-1} x_0^j A_2^{p-j} g_{n-2p+j}(x_1, x_2, x_3) + \sum_{j=p}^{n} x_0^j f_{n-j}(x_1, x_2, x_3) = 0,$$

where $n \geq 2p$, f_i and g_i are homogeneous polynomials, $g_{n-2p} \neq 0$, and $f_{n-p} \neq 0$.

A surface S_{2n} is *entirely spherical* if it contains the absolute conic as an n-fold curve. In this presentation we study the properties and visualize the shapes of entirely spherical surfaces which, in addition to the n-fold absolute conic, have singular n-fold points.

Key words: spherical surface, entirely spherical surface, Euclidean space, absolute conic, singular points, singular lines

MSC 2010: 51N20

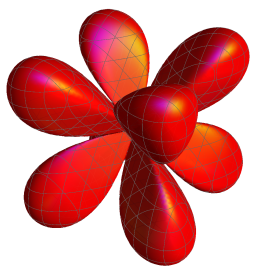

Figure 1: An example of a spherical surface

On some Geometrical Preserver Problems and their Stability

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Preserver problems can be roughly described as problems of finding the general form of all transformations on a given structure which preserve some property related to that structure. Furthermore, defining somehow the class of transformations that approximately preserve given property, stability problems concern the question of whether each transformation from this class can be approximated by an exact transformation preserving that property. The aim of this talk is to showcase some preserver problems arising from geometry, and to discuss their stability as well.

Key words: preservers, stability

MSC 2010: 47B49, 51F20, 51M05

Circular Curves in Euclidean Plane

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In the real projective plane the Euclidean metric defines the Euclidean plane with the *absolute* (*circular*) *points* $(0, 1, i)$ and $(0, 1, -i)$.

An algebraic curve passing through the absolute points is called *circular curve*. If it contains absolute points as its $p-$fold points, the curve is called $p-$*circular*. Every $p-$circular curve has the implicit equation in homogeneous coordinates of the following form:

$$\sum_{j=0}^{p-1} x_0^j (x_1^2 + x_2^2)^{p-j} g_{n-2p+j}(x_1, x_2) + \sum_{j=p}^{n} x_0^j f_{n-j}(x_1, x_2) = 0,$$

where g_k, $k = n-2p, ..., n-p-1$, and f_k, $k = 0, ..., n-p$, are homogeneous algebraic polynomials of degree k.
Obviously n must be at least $2p$. If $n = 2p$, the curve is called *entirely circular*.

We present some properties of circular curves and visualize their forms with the program *Mathematica*.

Key words: circular curves, Euclidean plane, absolute points

MSC 2010: 51N20

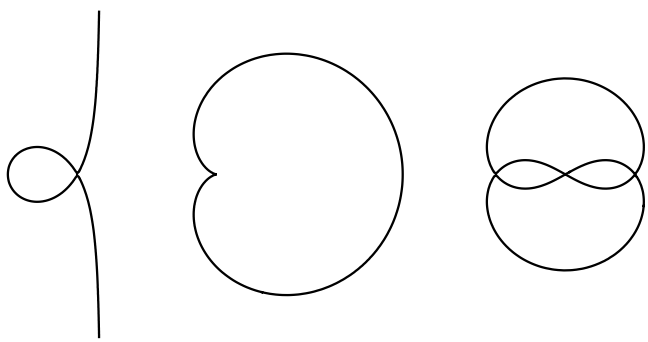

Figure 1: Examples of circular curves

Projective Models for Riemann Surfaces

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Riemann surfaces are one-dimensional complex manifolds, locally near every point they look like an open subset of the complex plane. They were introduced in 19th century by Riemann as a representation of algebraic curves. Theory of uniformization states that every Riemann surface of genus greater than 1 is a quotient of the complex upper-half plane by a Fuchsian group (discrete subgroup of $SL_2(\mathbb{R})$).

It is known that every Riemann surface can be embedded into projective space - such an embedding and its image are called a model for the Riemann surface. For a Riemann surface obtained as a quotient of a Fuchsian group Γ, one can construct a map into projective plane using modular forms of even weight on Γ. This is a holomorphic map and if its degree equals one it is birational equivalence.

Key words: Riemann surfaces, projective curves, uniformization theory

MSC 2010: 30F10 , 11F03 ,14H55

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Maclaurin Mapping in the pseudo-Euclidean Plane

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The presentation gives a new approach to pseudo-Euclidean plane (Minkowski) geometry based on *projective geometric algebra* (PGA) implementing both incidence and metric relations in a concise algebraic structure. It follows the projective (homogeneous) model embedding the pseudo-Euclidean plane $\mathbb{PE}_2$ in its projective extension, the real complexified projective plane $\mathcal{P}_2 = PG(2, \mathbb{R}) \subset PG(2, \mathbb{C})$, using 3-dimensional coordinates to model 2-dimensional plane, [1], [2].

After discussing the distinction between proper and ideal elements, a non-involutorial birational quadratic mapping named *Maclaurin mapping* will be studied in $\mathbb{PE}_2$, [6]. Depending on the position of the fundamental mapping elements, two types are to be distinguished in $\mathbb{PE}_2$, as well as in $\mathcal{P}_2$, each belonging to one of three possible plane Cremona mapping types, [3].

Also, types of entirely circular cubics obtained by Maclaurin mapping will be presented by using the pseudo-Euclidean interpretations of the projective situations, [4], [5].

Key words: projective geometric algebra (PGA), pseudo–Euclidean (Minkowski) plane, circular cubic, plane Cremona mapping

MSC 2010: 51A05, 51N25

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Implementation of Contemporary Methods in Teaching Descriptive Geometry at the Faculty of Civil Engineering and Architecture of Niš

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Since the foundation of the Faculty of Civil Engineering of Niš in 1965, the teaching of Descriptive Geometry has been performed in a classical way. Representation of three dimensional space on a two-dimensional medium, blackboard or paper was manual, both in lecturing and in exercise classes. The lectures provide fundamental knowledge, while the exercises provide step-by-step task solving, with additional clarifications. The curriculum remained the same in all those years. In order to facilitate work with the students and better grasp the curriculum, the teaching process was modernized:

- In 2009/10 the templates for the lectures were introduced, where the basic layout of the task was already drawn, to be completed by the students.
- In 2012 step-by-step graphic designs were made in Corel DRAW software, and all 13 presentations were placed on the Faculty website.
- In 2013, a free android application for step-by-step graphic designs was made, which could have been used by the students in the classes using their mobile phones.
- In the 2014/15 school year, a half of the total number of enrolled students attended the classical exercises, and the other half used the contemporary method of computer generated step-by-step graphic designs which were projected on a canvas screen.

This paper analyzes the impact of modernization of teaching on the students' success rate in passing the exams in the last 10 years at the study program of civil engineering. An analysis of the student passing the exam in the first four examination periods was performed and expressed in percents.
The results of the analysis conducted in the paper have indicated that the modernization of the teaching process contributed to the increase of the successfulness at the exams. The best results were exhibited by the students in the 2014/2015, which had at their disposal all the contemporary resources introduced to the teaching process.

Key words: descriptive geometry, classical method, contemporary method, step-by-step graphic design, android application

Kinematic Patterns of the Triangular Resch Tessellation

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In recent years origami tessellations have been investigated to exploit their kinematic properties in architecture. A particular kind of tessellation, called the triangular Resch tessellation, has caught the attention of many researchers. This tessellation is very flexible, but its degrees of freedom increase as it gets bigger. This creates problems for the management of its kinematic morphing.

Modeling and taking advantages of the kinematic principles of this and of other kinds of tessellation is not trivial and it needs advanced simulations. Understanding how the movement of one module spreads inside the tessellation could help predict and controlling the response of the whole tessellation in both the simulated and real space.

This study wants to find out a set of kinematic schemes which restrict the liberty of some parts of the tessellation in order to obtain new kinds of behavior of the origami.

To achieve this aim, the tessellation will be considered as a group of modules. For some modules a set of rules that will affect the degrees of freedom of the whole tessellation will be established. To find which modules we have to constrain and which rules we have to apply to them, a visual-logic process is proposed to create a kinematic pattern that will limit the liberty of the tessellation. Computer based digital simulation will be used to verify that the modified kinematic behavior of the tessellation will be functioning.

In conclusion, several patterns with different targets are proposed to reduce the kinematic uncertainty of the triangular Resch tessellation. They can be applied to the modeling phase to obtain a better control of the simulation, or in the realization of a real kinematic origami structure to minimize the number of controlled modules. In future works, it could be possible to apply this method to other kinds of tessellation.

Key words: origami, tessellation, kinematics, rigid body

MSC 2010: 70B10

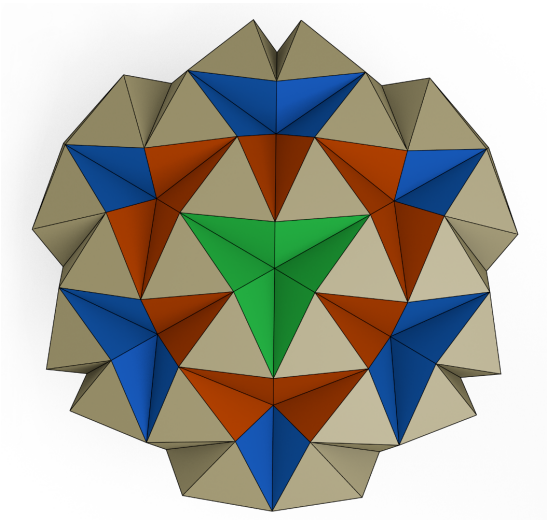

Figure 1: The color of the faces indicates the value of the dihedral angles.

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Hyperbolic Pascal Triangles

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There are several approaches to generalize Pascal's arithmetic triangle. In the lecture, we introduce a new generalization of Pascal's triangle. The new object is called the hyperbolic Pascal triangle since the mathematical background goes back to regular mosaics on the hyperbolic plane. We describe the procedure of how to obtain a given type of hyperbolic Pascal triangle from a mosaic denoted by Schläfli's symbol $\{p, q\}$, where $(p-2)(q-2) > 4$. Figure 1 illustrates the hyperbolic Pascal triangle when $\{p, q\} = \{4, 5\}$.

Then we study certain quantitative properties such as the number, the sum, and the alternating sum of the elements of a row. Moreover, the pattern of the rows, and the appearance of some binary recurrences in a fixed hyperbolic triangle are investigated.

Key words: Pascal triangle, regular mosaics on hyperbolic plane

MSC 2010: 11B99, 05A10

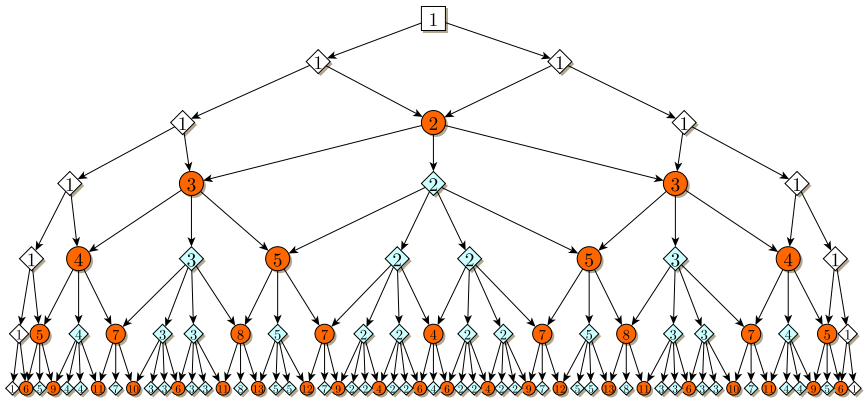

Figure 1: Hyperbolic Pascal triangle linked to $\{4, 5\}$ up to row 6

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Augmented Reality Presentation of Geometrical Surfaces in Architecture

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Architects must pay attention to the form (shape) of the object, during the building design process. At the same time architects should be familiar with geometric shapes that can be used, in order to create more attractive object design [1]. For this reason, during the education of architectural students, attention must be paid to geometrical surfaces and their application in architecture. The traditional method of education involves the use of a book in which the geometric surface is presented by using two-dimensional images and text explanations. Main problem of traditional education is presentation of complex three-dimensional surfaces on two-dimensional paper. For a better understanding it is necessary to present three-dimensional spatial models of geometric surfaces. For this purpose, it is possible to apply a method of augmented reality by which the user's perception of the real world is supplemented with virtual objects [2]. Android application for the presentation of three-dimensional models of geometric surfaces using augmented reality method is created and presented under the scope of this paper. The photos in book "Geometric Surfaces in Architecture" [1] are used as a basis and indicated as markers for virtual 3D model presentation. Corresponding mobile application has been created to enable the user to view the book using android device with additional 3D content (Figure 1). The use of the application is simple, while the overview of geometrical surfaces 3D models is completely intuitive. This approach provides easier learning and better understanding of geometrical surfaces by using a contemporary digital technology.

Key words: augmented reality, geometrical surfaces, architecture, buildings

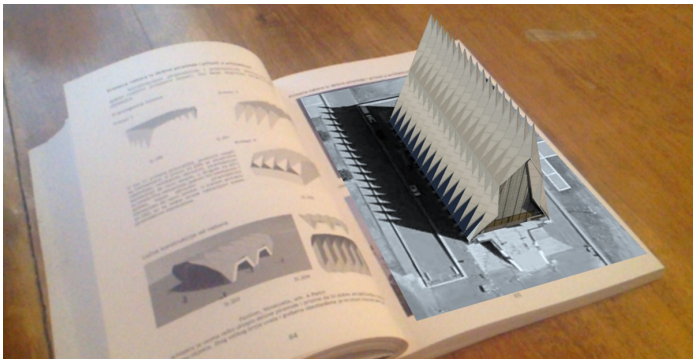

Figure 1: Augmented reality presentation of pleated surface building

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Modern 3D Technology and Geometry

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Engineers of today are using their geometric competences in a new technological environment. For example, instead of measuring and drawing current situation before doing project design, engineers use 3D scanning. They also use a variety of modelling methods and 3D models when designing buildings. Prototyping and reverse engineering are used also. New technology allows all necessary 2D views to be obtained from 3D models automatically.

Some of these new skills are included in geometric e-learning courses at Faculty of Civil Engineering (GF) in Rijeka (object modelling, automatic views and sections, unrolling of surfaces ...) Others will be included later.

New technological environment is used in geometric research also.

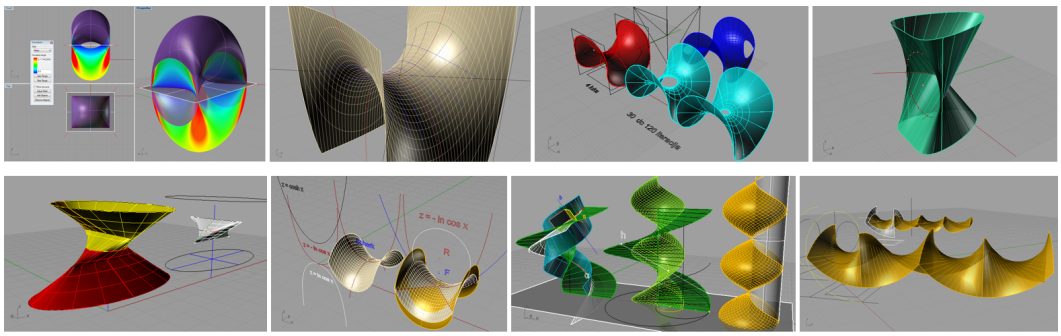

Figure 1: Examples of curvature analysis, geodesic lines on minimal surface, iterative form finding, constructive elaboration of ruled quartics, translation surfaces, etc.

New technologies in engineering and architecture include laser 3D scanning process, data processing, modelling and 3D printing (prototyping).

GF Rijeka has recently purchased *"FARO Focus 3D"*, a 3D laser scanner. It is already used on two projects. Within the first project, theatre building *"Teatro Fenice"* is scanned, and within the second project, rocky coast of the island of Krk (in Baška) is scanned. Each 3D image consists of several million points. The integrated camera takes photo-realistic color scans. Scanned data is processed by *"SCENE"*, a point cloud software that allows not only 3D visualisation but also meshing and exporting in various data formats for further processing.

The point cloud can also be imported into a CAD software and used as a basis for modelling. Result can be evaluable on web as a digital 3D model of the object.

For purposes of this talk, I used Rhino for processing point cloud.

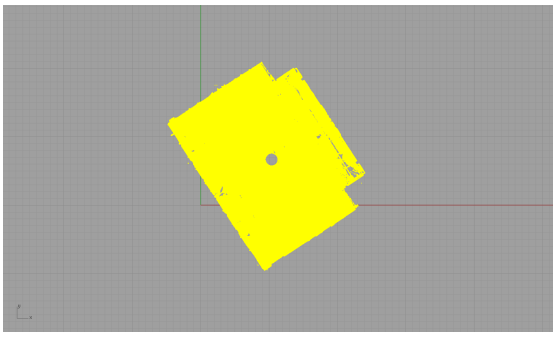

Figure 2: Point cloud of *"Teatro Fenice"* roof imported in Rhino

There are several reverse engineering plug-ins for Rhino. Some of them are *RhinoResurf* and *RhinoReverse*. Other tools for reverse engineering are *Flexicad*, *Geomagic*...).

GF Rijeka has also recently purchased a *"Connex500"* multi-material 3D printing system. It allows simulation of the final product by combining multiple materials with varied properties and tones. It can accept models in `STL` and `SLC` formats, which are both supported by Rhino.

Key words: modelling, new technology, point cloud, 3D scanner, 3D printer

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Harmonic Evolutes of Timelike Ruled Surfaces in Minkowski Space

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Let S be a ruled surface in 3-dimensional Minkowski space $\mathbf{R}^3_1$ parameterized by

$$\mathbf{x}(u,v) = c(u) + ve(u),$$

where $c(u)$ is a base curve and $e(u)$ a non-vanishing vector field along c which generates the rulings. Ruled surfaces in Minkowski space $\mathbf{R}^3_1$ are classified with respect to the casual character of their rulings which can be either space-like, time-like or null (light-like). A time-like ruled surface inherits the pseudo-Riemannian metric of index 1 from the ambient space. It is generated in the following cases: when c is a space-like curve and $e(u)$ a time-like field (then $e'(u)$ is space-like) or vice-versa, when c is a time-like curve and $e(u)$ a space-like field (with $e'(u)$ either null or non-null). It is also generated when $c'(u)$, $e(u)$ are both null. The last ruled surfaces are called the null-scrolls, or in the special case, the B-scrolls. In this presentation we investigate properties of harmonic evolutes of time-like ruled surfaces in Minkowski space. The harmonic evolute of a surface is the locus of points which are harmonic conjugates of a point of a surface with respect to its centers of curvature p_1 and p_2. Basic properties of harmonic evolutes of surfaces in Euclidean space have been investigated in [1] and of surfaces in Minkowski space in [2].

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Strophoids and the Equicevian Points of a Triangle

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A point P in the plane of a given triangle is said to be equicevian if the three cevians through P are of equal length. Equicevian points outside the sidelines of the triangle are called proper. They can be obtained by intersecting three cubic curves, so-called equicevian cubics. These curves are strophoids, i.e., circular cubics with a node and with orthogonal tangents at the node. It will be proved that in the Euclidean plane the proper equicevian points are the real and the imaginary focal points of the Steiner circumellipse. The lecture concludes with a medley of geometric problems where strophoids show up as a geometric locus.

Laces

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A family of curves named "Laces" is introduced and investigated, while some of their intrinsic geometric properties are derived and presented. Particular representatives of this two-parametric family of curves can be generated by means of Minkowski summative or Minkowski multiplicative combinations of two equally parameterized curve segments in the Euclidean space $\mathbf{E}^n$.

Let two curve segments be determined by respective vector maps defined on the same interval $I \subset \mathbf{R}$

$$K : \mathbf{rk}(u) = (xk_1(u), xk_2(u), ..., xk_n(u))$$

$$L : \mathbf{rl}(u) = (xl_1(u), xl_2(u), ..., xl_n(u)).$$

Minkowski summative combination of curves K and L

$$S = a.K \oplus b.L, a, b \in \mathbf{R}$$

is a family of curve segments in $\mathbf{E}^n$ parametrically represented on $I \subset \mathbf{R}$ by vector maps

$$S : \mathbf{s}(u) = a.\mathbf{rk}(u) + b.\mathbf{rl}(u) = (xs_1(u), xs_2(u), ..., xs_n(u)),$$

where $xs_i(u) = a.xk_i(u) + b.xl_i(u)$, for $i = 1, 2, ..., n$.

Differential characteristics of the two-parametric family of summative laces can be derived and represented by means of derivatives of vector representations of the two operand curves.

Minkowski multiplicative combination of curves K and L

$$P = a.K \otimes b.L, a, b \in \mathbf{R}$$

is a family of curve segments in $\mathbf{E}^d, d = n(n-1)/2$, parametrically represented on $I \subset \mathbf{R}$ by vector maps

$$P : \mathbf{p}(u) = a.\mathbf{rk}(u) \wedge b.\mathbf{rl}(u) = (xp_1(u), xp_2(u), ..., xp_d(u)),$$

where the following relations hold for coordinate functions

$$xp_k(u) = ab(xk_i(u)xl_j(u) - xk_j(u)xl_i(u))i(u)),$$

for all combinations of pairs of coefficients $i, j = 1, 2, ..., n, i \neq j$, while $k = 1, 2, ..., d$.

Differential characteristics of the two-parametric family of multiplicative laces can be also derived from derivatives of vector maps of the two respective curves.

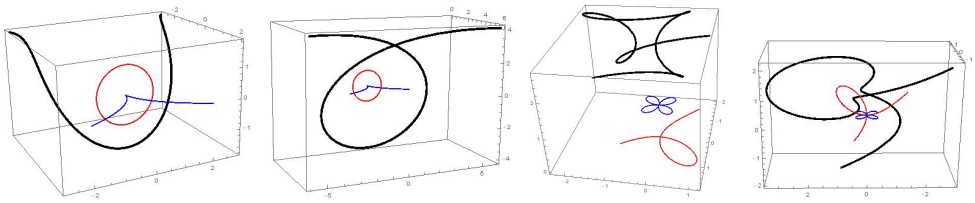

Figure 1: Minkowski summative combinations of two curve segments

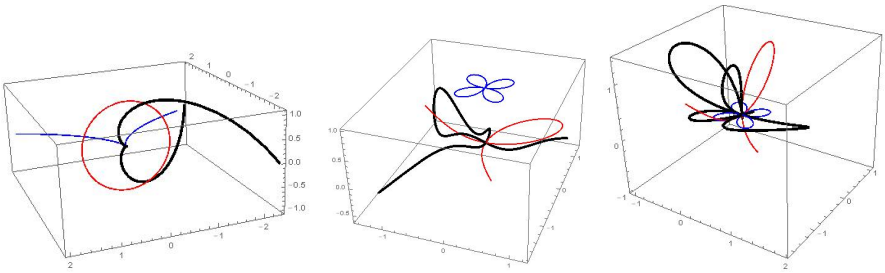

Figure 2: Minkowski multiplicative combinations of two curves segments

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Elementary Constructions for Conics in Hyperbolic and Elliptic Planes

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In the Euclidean plane there are well-known constructions of points and tangents of, e.g., an ellipse c based on the given axes of c, which make use of the Apollonius definition of c via its focal points or via two perspective affinities (de la Hire's construction). Even there is no parallel relation neither in a hyperbolic plane nor in an elliptic plane, it is possible to modify many of the elementary geometric constructions for conics, such that they also hold for those (regular) non-Euclidean planes. Some of these modifications just replace Euclidean straight lines by non-Euclidean circles. Furthermore we also study properties of Thales conics, which are generated by two pencils of (non-Euclidean) orthogonal lines.

Key words: hyperbolic plane, elliptic plane, conic sections, de la Hire, Apollonius, Thales

MSC 2010: 51M04, 51M09

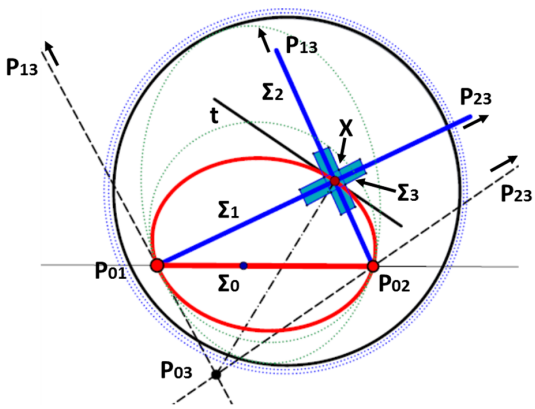

Figure 1: Thales conic and its kinematic construction of points and tangents in a hyperbolic plane

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Five Cubes in the Dodecahedron and 3D Models of Hypercubes

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The 3-dimensional framework (3-model) of any k-dimensional cube (k-cube) can be produced based on initial k edges arranged by rotational symmetry, whose Minkowski sum can be called zonotope. Combining $2 < j < k$ edges, 3-models of j-cubes can be built, as parts of a k-cube. The suitable combinations of these zonotope models can result in 3-dimensional space-filling mosaics. The investigated periodical tessellations always hold the 3-model of the k-cube and necessary j-cubes derived from it. The intersections of the mosaics with planes allow unlimited possibilities to produce periodical symmetric plane-tiling. Moving intersection planes results in series of tessellations or grid-patterns transforming into each other. These can be shown in varied animations.

Five cubes can be constructed in the Platonic dodecahedron with joining vertices. The common part of these cubes is the rhombic triacontahedron, hull of a 3-model of the 6-cube. Always three edges, perpendicular to each other, may be chosen from the five cubes. These can be the initial edges of 3-models of the 6-, 9-, 12-, 15-cubes. So the edges of these special models join edges of common cubes. Each such model with their derived lower-dimensional parts makes possible to construct periodical space-filling mosaics easier because of the special symmetry properties of the models. The hull of the considered 3-model of the 15-cube is the Archimedean truncated icosidodecahedron. The above models are lower-dimensional parts of this one.

The question of the relating paper from 2011 is answered. The third of the Archimedean solids which are hull of a 3-model of a hypercube can be also the initial stone of a periodical space-filling mosaic.

Key words: constructive geometry, hypercube modeling, tessellation

MSC 2010: 52B10; 52B12, 52B15, 65D17

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Cyclic Quadrilaterals, Formulas of Brahmagupta and Parameshvara, and One-Dimensional Rational Trigonometry

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The classical formulas of Brahmagupta and Parameshvara for the area and circumradius of a cyclic quadrilateral can be reformulated into a purely algebraic form, which allows them to apply equally to the convex and non convex cases, generalize to arbitrary fields, and extend also to relativistic geometry. The key is to understand how they connect with fundamental formulas from rational trigonometry, both in the affine and projective one-dimensional settings.

Posters

The Rosetta Project - an Exploration Into Geometric Polymorphism

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A 3D model called Rosa Mystica, made as a geometric representation of an abstract flower, was designed by the geometry generation on top of a hexagonal base. Using parametric modelling, this model was translated into one simple morphogenetic formula. The initial intention was to create multiple versions of Rosa Mystica but it has turned into an art project of its own, The Rosetta Project. The Rosetta Project represents a top/bottom parallel view projection of a 3D aggregation of tetrahedra geometry generated upon an n-sided pyramid. The initial (Rose) hexagon base was replaced with a parametric polygon. The polygon is placed in the origin of the coordinate system and it is defined by the parameters of the radius of the circumscribed circle, and the number of sides. The third parameter determines the height of the pyramid raised on top of the polygon (faces are extruded to a point above its centroid point). The height of all other extrusions, which are raised on top of the offset faces of an exploded geometry, is the same as the offset distance. The operation is repeated three more times.

Parameters:

1. Initial radius of circumscribed polygon,
2. Number of polygon sides,
3. Height of the initial pyramid,
4. Offset distance.

Construction of the 3D model for the rosetta:

1. Draw a polygon in the coordinate origin,
2. Raise a pyramid on top of the polygon,
3. Explode the pyramid,
4. Offset exploded faces outwards,
5. Raise a pyramid on top of every offset face,
6. Repeat steps form 3. to 5. three more times.

Although the formula for generating rosettas is very simple, the final geometry is complex. Some rosettas seem to be flat, some relief, deep, sharp, extroverted, closed ... This way polymorphic results are sometimes similar to living organisms, such as plants or microbes, and sometimes they are rather alien and crystal like. A simple formula produced geometric polymorphism that is abstract enough for aesthetic purposes but also bio-like enough to resemble natural polymorphisms.

Key words: generative geometry, parametric modelling, polymorphism, orthogonal projection, art

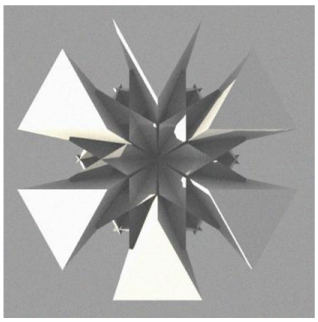

Figure 1: Rosa Mystica, the proto-Rosetta

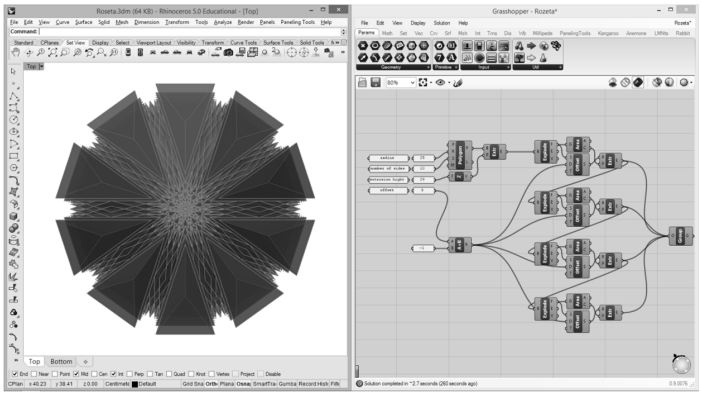

Figure 2: Screenshot of generative algorithm for the Rosetta (right) and corresponding generated geometry (left)

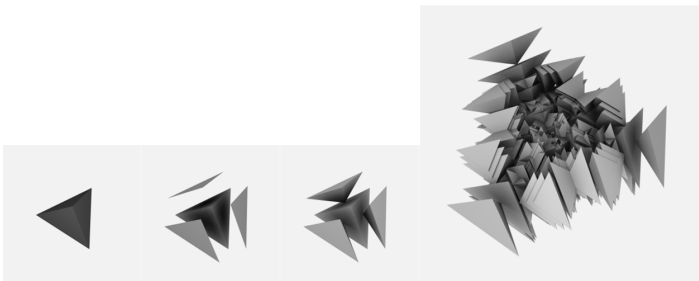

Figure 3: Step 3 to 5 of the 3D geometry generation (above) and final 3D geometry (right)

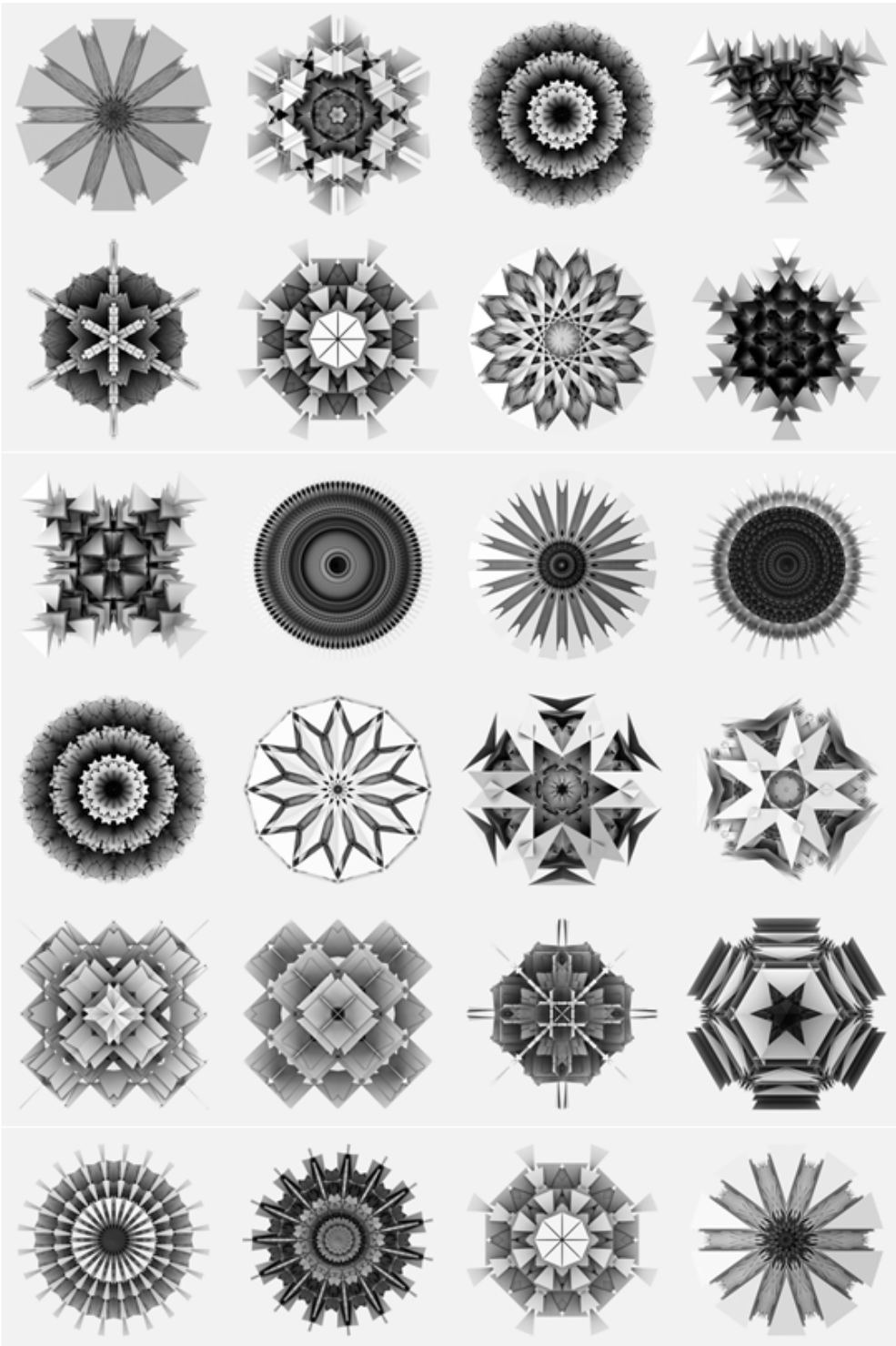

Figure 4: Some of the infinite shapes generated by a simple and deterministic formula

A Sequel to "Interaction Among Courses"

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The scientific and professional cooperation among geometry teachers at the technical faculties in Croatia in the early nineties was encouraged by the founding of the Croatian Society for Geometry and Graphics. Since then, this cooperation has resulted in a number of projects.
The authors were additionally motivated through their work on the project "GeomTeh3D", approved as a development project and supported by the Fund for the Development of the University, University of Zagreb, for the 2011/2012 academic year. The objectives of the project, among others, were to enhance further the collaboration among the teachers of mathematics/geometry courses, and to improve teaching methodologies.
One of the results in this respect is the preparation of "joint" educational posters. The first such poster we presented at the 16th International Conference on Geometry and Graphics held in Innsbruck, Austria, in 2014. As we announced we have continued our work the result of which will be shown at 18th Scientific-Professional Colloquium on Geometry and Graphics in Beli Manastir.

Key words: mathematics education, interaction

MSC 2010: 97B40, 97G80, 97G70, 97H60

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Circular Curves of the 3rd Class Obtained by Pedal Transformation in the quasi-Hyperbolic Plane

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In the quasi-hyperbolic plane the pedal transformation with respect to a given polar line p maps a point T into the pedal line which passes through the point T and its perpendicular point incident with the polar line p. The pedal curve c_p of a given curve c with respect to the polar line p is the locus of the pedal lines of all the points of the curve c. Different types of circular curves of the 3rd class obtained as the pedal curves of conics will be shown.

Key words: pedal transformation, pedal curve, quasi-hyperbolic plane

MSC 2010: 51A05, 51M15, 51N25

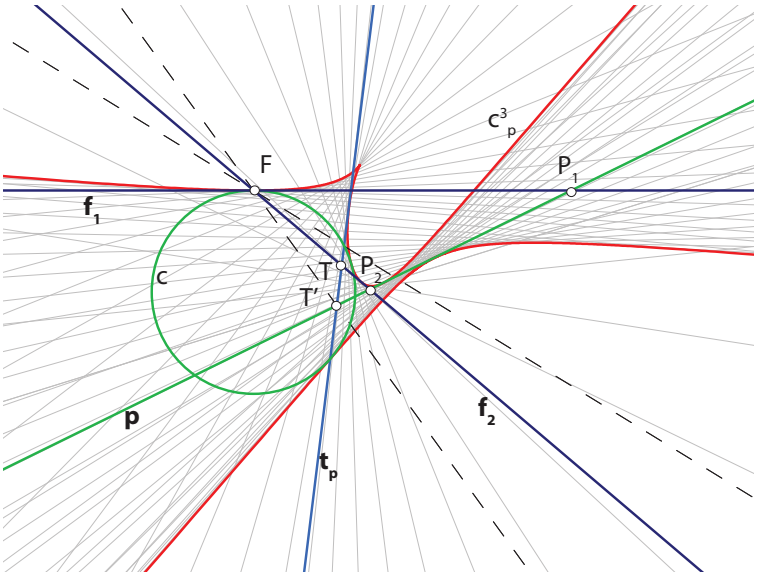

Figure 1: Circular curve c_p^3 of type (2,1) obtained as the pedal curve of the special parabola c

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On Certain Classes of Weingarten Surfaces in *Sol* Space

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A Weingarten surface is a surface satisfying the Jacobi equation

$$\Phi(K,H) = \det \begin{pmatrix} K_u & K_v \\ H_u & H_v \end{pmatrix} = 0,$$

where K is Gaussian curvature and H is mean curvature of the surface.

The study of Weingarten surfaces was initiated by J. Weingarten in 1861. E. Beltrami and U. Dini few years later proved that the only non-developable Weingarten ruled surface in Euclidean 3-space is a helicoidal ruled surface. In the last decade several papers on Weingarten surfaces in different 3-dimensional spaces have appeared (see [2], [3] and [6]).

The *Sol* geometry is one of the eight homogeneous Thurston 3-geometries. More about curves and surfaces in *Sol* geometry can be found in [1], [4] and [5]. Motivated by the fact that there are no results about Weingarten surfaces in *Sol* geometry, we examine some classes of ruled Weingarten surfaces in *Sol* geometry.

Key words: *Sol* geometry, Weingarten surface, ruled surface

MSC 2010: 53C30, 53B25

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Presentation of Old Building Facade Using Augmented Reality

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Construction work on the facade of an object over time can positively or negatively affect the architectural value of a building. In Serbia there are numerous examples of negative changes of facade geometry in the period following the Second World War. Facade decoration of many important buildings from the period of classicism, secession and early modernism was completely removed, so that today's appearance can be hardly connected to the original [1]. In this paper, the process of digital reconstruction and presentation of the former building facade on the corner of Generala Milojka Lešjanina street and Milorada Veljkovića Špaje street in Niš, Serbia, is presented. Method of semiautomatic photogrammetry is used for 3D model creation of the former facade geometry and presented using augmented reality. Augmented reality is an emerging computer technology where the perception of the user is enhanced by the seamless blending between a realistic environment and computer-generated virtual object coexisting in the same space [2].

Key words: old facade, photogrammetry, augmented reality, 3D model, architecture

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Exhibition

Magical Dimensions

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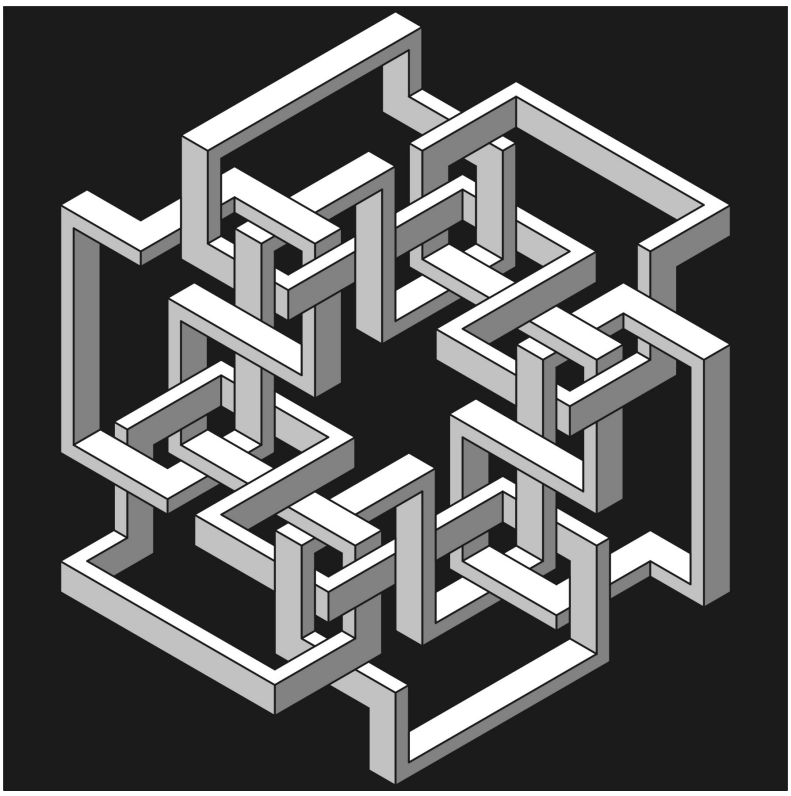
The exhibition introduces a selection from the investigations of the last few years.



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